6. Discrete Linear Stochastic Processes/ Models

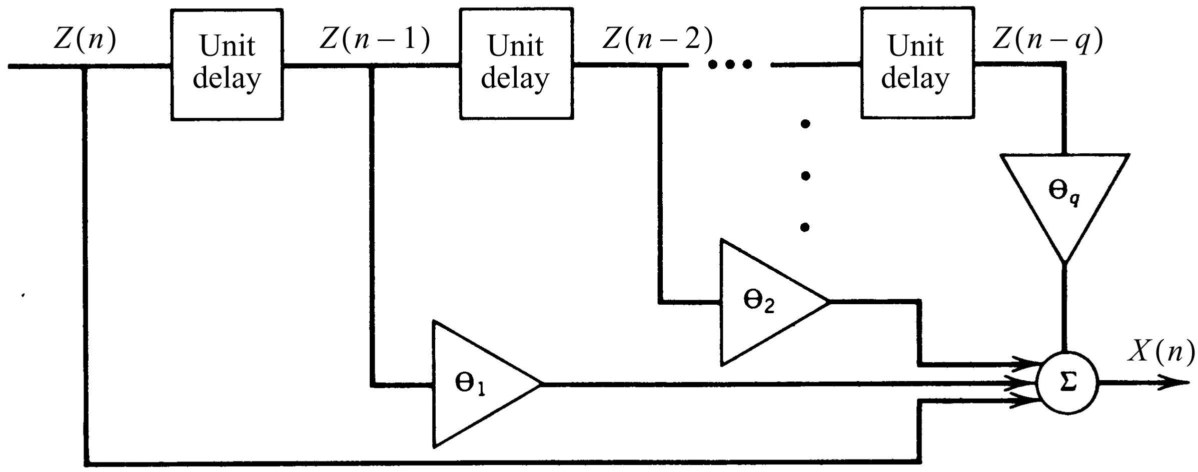
6.1. Moving average processes

- ***Definition:***
  A random sequence $X(n)$ is a moving average process of order q (MA(q)) if for any n:

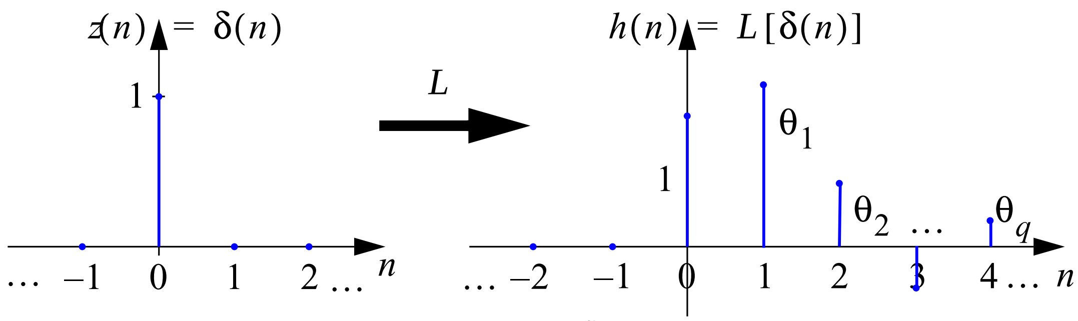
$$X(n) = Z(n) + \sum_{i=1}^{q} \theta_i Z(n-i)$$

  where $Z(n)$ is a white Gaussian process.

- ***Transversal filter implementation of a MA(q) process:***



- ***Impulse response of the transversal filter:***



$$h(n) = \delta(n) + \sum_{i=1}^{q} \theta_i \delta(n-i)$$

- ***Stability and causality:***
  Transversal filters are stable and causal.

- ***Transfer function of the transversal filter:***

$$H(f) = 1 + \sum_{i=1}^{q} \theta_i \exp(-j2\pi i f)$$

Proof:

$$x(n) = \sum_{i=1}^{q} \theta_i z(n-i) + z(n)$$

$$X(f) = \sum_{i=1}^{q} \theta_i \exp(-j2\pi i f) Z(f) + Z(f)$$

$$= \left[1 + \sum_{i=1}^{q} \theta_i \exp(-j2\pi i f)\right] Z(f)$$

□

- ***Power spectrum of a MA(q) process:***

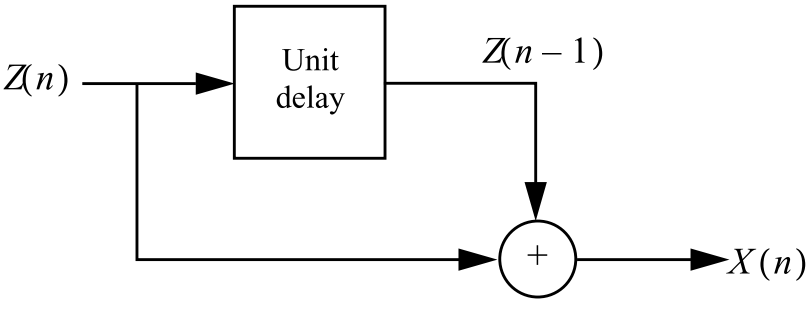
$$S_{XX}(f) = \left|1 + \sum_{i=1}^{q} \theta_i \exp(-j2\pi i f)\right|^2 \sigma_Z^2$$

- ***Mean value and autocorrelation function of a MA(q) process:***

$$\mu_X = 0$$

$$R_{XX}(k) = \sigma_Z^2 R_{hh}(k)$$

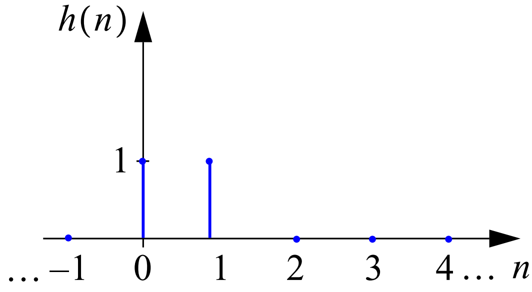
- ***Example: MA(1)***



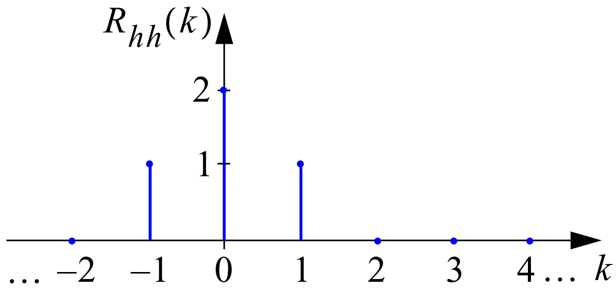
$$X(n) = Z(n) + Z(n-1) \qquad (\theta_1 = 1)$$

- Impulse response and autocorrelation function of the transversal filter

$$h(n) = \begin{cases} 1 ; & n \in \{0, 1\} \\ 0 ; & \text{elsewhere} \end{cases}$$



$$R_{hh}(k) = \begin{cases} 2 ; & k \in \{0\} \\ 1 ; & k \in \{-1, 1\} \\ 0 ; & \text{elsewhere} \end{cases}$$

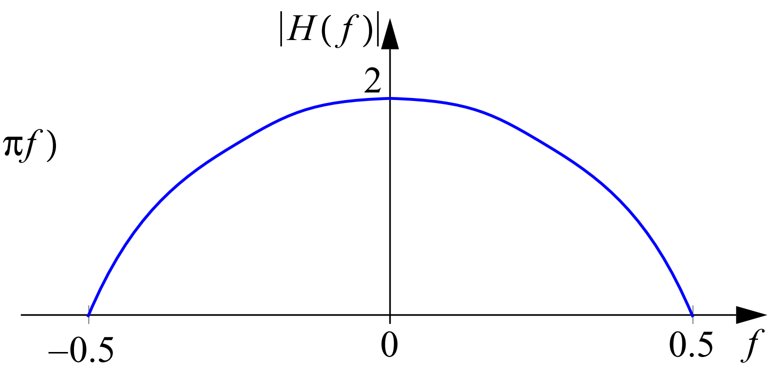


- Transfer function:

$$\begin{aligned} H(f) &= 1 + \exp(-j2\pi f) \qquad |f| \le 0.5 \\ &= \exp(-j\pi f)[\exp(j\pi f) + \exp(-j\pi f)] \\ &= 2\exp(-j\pi f)\cos(\pi f) \end{aligned}$$

$$|H(f)| = 2\cos(\pi f)$$

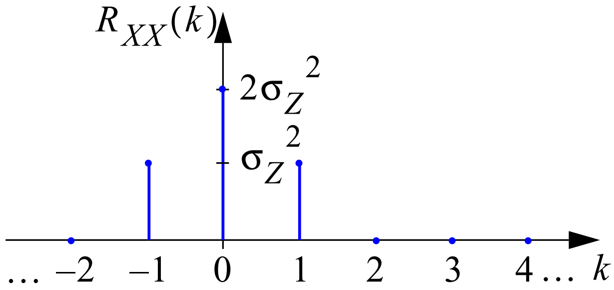
$$(|f| \le 0.5)$$



- Autocorrelation function of $X(n)$:

$$R_{XX}(k) = \sigma_Z^2 R_{hh}(k)$$

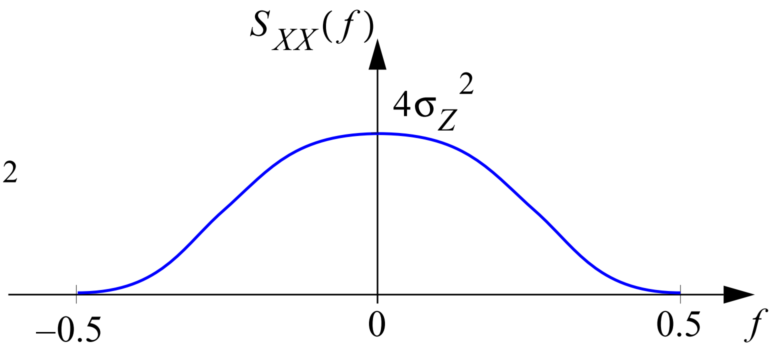
$$= \begin{cases} 2\sigma_Z^2 & ; \quad k \in \{0\} \\ \sigma_Z^2 & ; \quad k \in \{-1, 1\} \\ 0 & ; \quad \text{elsewhere} \end{cases}$$



- Power spectrum of $X(n)$:

$$S_{XX}(f) = \sigma_Z^2 |H(f)|^2$$

$$= 4\sigma_Z^2 \cos(\pi f)^2$$



6.2. Autoregressive processes

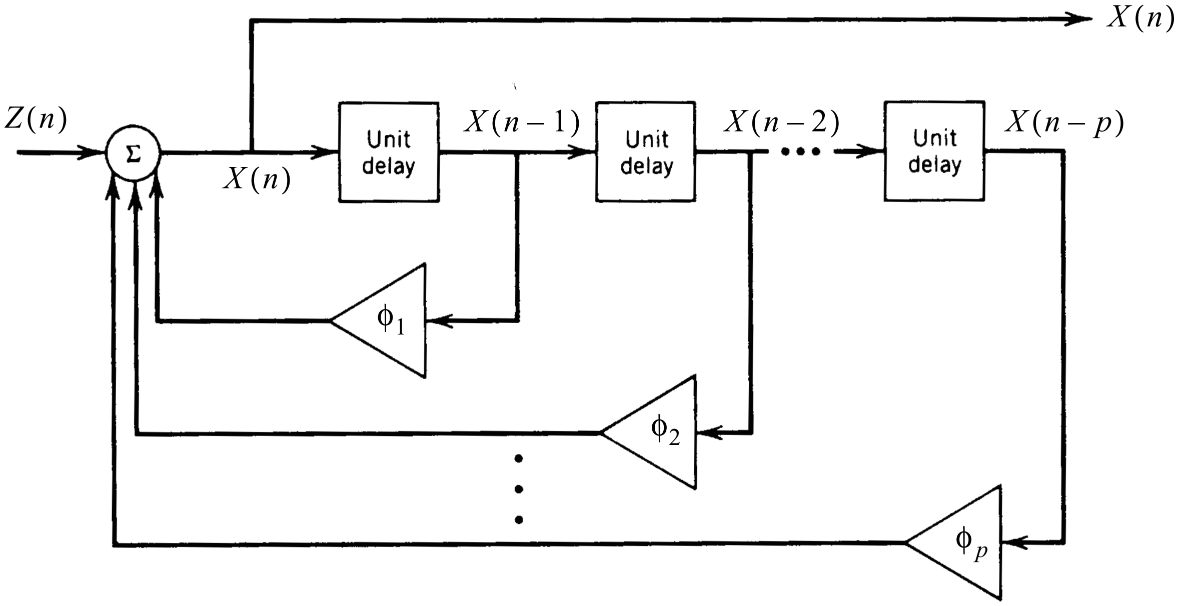
- ***Definition:***

  A random sequence $X(n)$ is an autoregressive process of order p (AR(p)) if it is WSS and for any n:

$$X(n) = \sum_{i=1}^{p} \phi_i X(n-i) + Z(n)$$

  where $Z(n)$ is a white Gaussian process.

- ***Recursive filter implementation:***



- ***Causal and stable AR processes:***

  An AR(p) process $X(n)$ is called causal and stable if there exists an infinite causal and stable transversal filter with impulse response $h(n)$ such that

$$X(n) = \sum_{i=0}^{\infty} h(i)Z(n-i)$$

$$= h(n)*Z(n)$$

  Let us define the polynomial

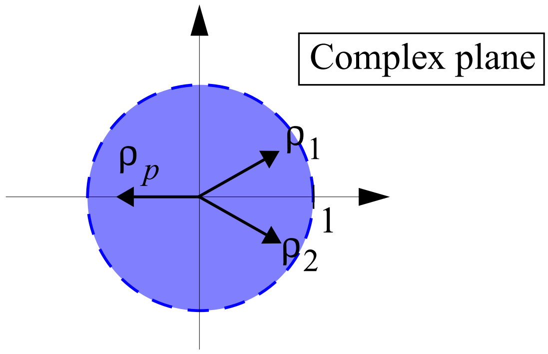
$$\phi(z) \equiv 1 - \sum_{i=1}^{p} \phi_i z^{-i} \qquad z \text{ : complex variable.}$$

  Then, the AR process $X(n)$ is causal and stable, if, and only if, the roots of $\phi(z)$ are located inside the unit circle, i.e. if $\phi(z)$ factorizes according to

$$\phi(z) = \prod_{i=1}^{p} (1 - \rho_i z^{-i})$$

  with $|\rho_i| < 1$, $i = 1, \ldots, p$.

Location of the roots of $\phi(z)$ in the complex plane when $X(n)$ is causal and stable:



The impulse response of a causal and stable AR(p) process is determined by the identity

$$\sum_{i=0}^{\infty} h(i)z^{-i} = \frac{1}{\phi(z)} \qquad |z| \geq 1$$

- ***Transfer function of the recursive filter:***

$$H(f) = \frac{1}{1 - \sum_{i=1}^{p} \phi_i \exp(-j2\pi i f)}$$

Proof:

$$x(n) = \sum_{i=1}^{p} \phi_i x(n-i) + z(n)$$

$$X(f) = \sum_{i=1}^{p} \phi_i \exp(-j2\pi i f) X(f) + Z(f)$$

$$= \left[\sum_{i=1}^{p} \phi_i \exp(-j2\pi i f)\right] X(f) + Z(f) \qquad \square$$

- ***Power spectrum of an AR(p) process:***

$$S_{XX}(f) = \frac{\sigma_Z^2}{\left|1 - \sum_{i=1}^{p} \phi_i \exp(-j2\pi i f)\right|^2}$$

- ***Mean value and autocorrelation function of a causal AR(p) process:***
  If the AR process $X(n)$ is causal,

$$\mu_X = 0$$
$$R_{XX}(k) = \sigma_Z^2 R_{hh}(k)$$

- ***Example: AR(1):***
  The first-order recursive filter discussed in the previous chapter with a white Gaussian process as the input signal generates an AR(1) process.
- ***Yule-Walker equations:***
  Let be $k \geq 0$:

$$X(n) = \sum_{i=1}^{p} \phi_i X(n-i) + Z(n)$$

$$X(n)\, X(n-k) = \sum_{i=1}^{p} \phi_i X(n-i) X(n-k) + Z(n) X(n-k)$$

$$\mathbf{E}[X(n)\, X(n-k)] = \sum_{i=1}^{p} \phi_i \mathbf{E}[X(n-i) X(n-k)] + \mathbf{E}[Z(n) X(n-k)$$

$$R_{XX}(n, n-k) = \sum_{i=1}^{p} \phi_i R_{XX}(n-i, n-k) + \sigma_Z^2 \delta(k)$$

$$R_{XX}(k) = R_{XX}(-k) = \sum_{i=1}^{p} \phi_i R_{XX}(i-k) + \sigma_Z^2 \delta(k)$$

Using a vector notation, for $0 \leq k \leq p$

$$R_{XX}(k) = [R_{XX}(1-k), \ldots, R_{XX}(p-k)] \begin{bmatrix} \phi_1 \\ \ldots \\ \phi_p \end{bmatrix} + \sigma_Z^2 \delta(k) \qquad (6.1)$$

For $k > p$:

$$R_{XX}(k) = [R_{XX}(k-1), \ldots, R_{XX}(k-p)] \begin{bmatrix} \phi_1 \\ \ldots \\ \phi_p \end{bmatrix} \qquad (6.2)$$

Let us define

$$\Phi \equiv \begin{bmatrix} \phi_1 \\ \dots \\ \phi_p \end{bmatrix} \qquad \gamma \equiv \begin{bmatrix} R_{XX}(1) \\ R_{XX}(2) \\ \dots \\ R_{XX}(p) \end{bmatrix}$$

$$\Gamma \equiv \begin{bmatrix} R_{XX}(0) & R_{XX}(1) & \dots & R_{XX}(p-1) \\ R_{XX}(-1) & R_{XX}(0) & \dots & R_{XX}(p-2) \\ \dots & \dots & \dots & \dots \\ R_{XX}(-(p-1)) & R_{XX}(-(p-2)) & \dots & R_{XX}(0) \end{bmatrix}$$

Note that Γ is symmetric.

Then, for $k = 0$ Identity (6.1) becomes

$$\boxed{R_{XX}(0) = \gamma^T \Phi + \sigma_Z^2}$$

Inserting $k = 1, \dots, p$ in (6.1) yields p identities that can be concatenated in a matrix form according to

$$\begin{bmatrix} R_{XX}(1) \\ R_{XX}(2) \\ \dots \\ R_{XX}(p) \end{bmatrix} = \begin{bmatrix} R_{XX}(0) & R_{XX}(1) & \dots & R_{XX}(p-1) \\ R_{XX}(-1) & R_{XX}(0) & \dots & R_{XX}(p-2) \\ \dots & \dots & \dots & \dots \\ R_{XX}(-(p-1)) & R_{XX}(-(p-2)) & \dots & R_{XX}(0) \end{bmatrix} \begin{bmatrix} \phi_1 \\ \phi_2 \\ \dots \\ \phi_p \end{bmatrix}$$

$$\boxed{\gamma = \Gamma\Phi}$$

Comments:

- The feed-back coefficients $\phi_1, \dots, \phi_p$ of the recursiv filter and the variance σ_Z^2 of the white Gaussian input process $Z(n)$ can be computed from $R_{XX}(0), \dots, R_{XX}(p)$ via the Yule-Walker equations and vice-versa.
- The samples $R_{XX}(k)$, $k > p$ can be recursively computed from $\phi_1, \dots, \phi_p$ and $R_{XX}(k-1), \dots, R_{XX}(k-p)$ by using Identity (6.2).

6.3. Autoregressive moving average processes

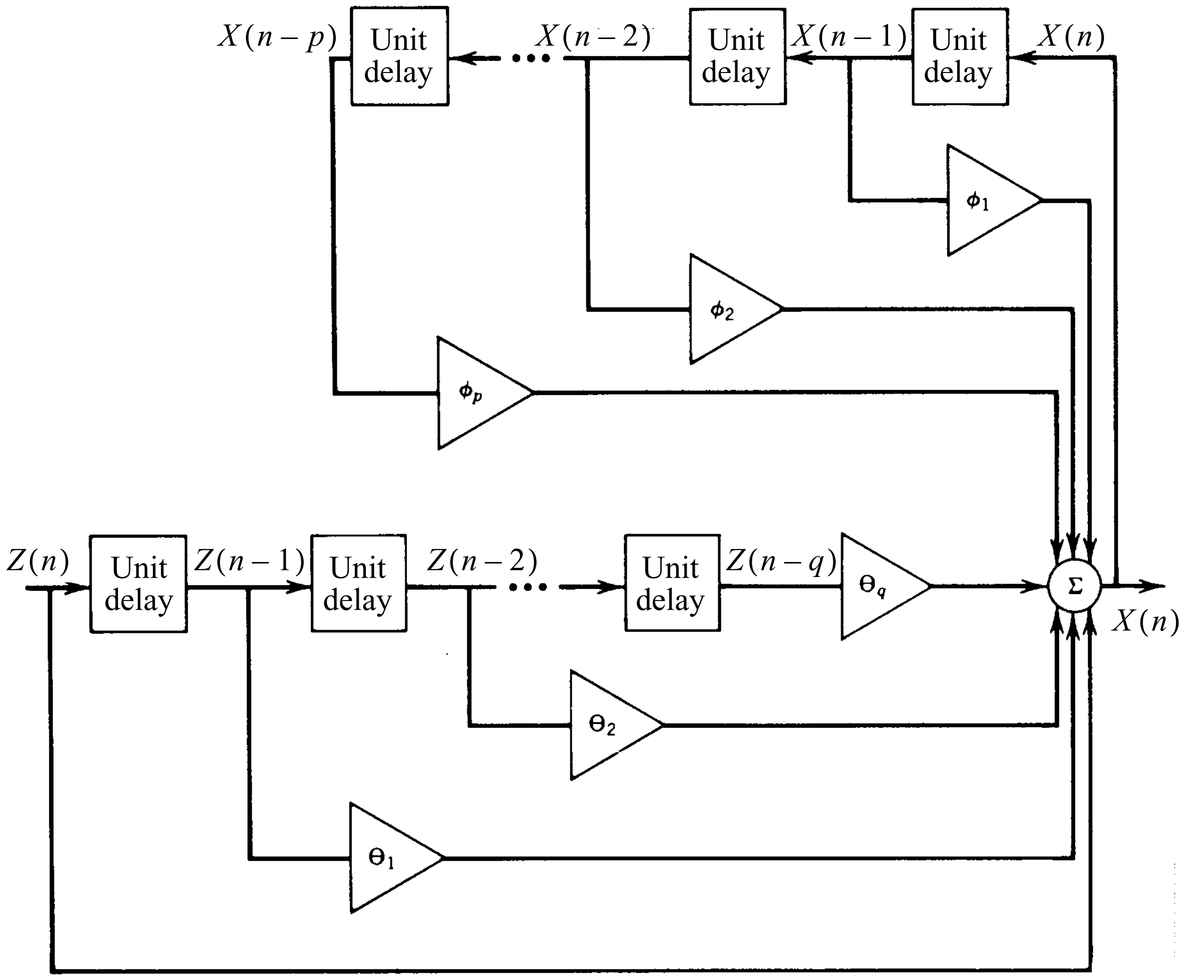
- ***Definition:***

  A random sequence $X(n)$ is an autoregressive moving average process (p, q) th order (ARMA((p, q))) if it is WSS and for any n:

$$X(n) = \sum_{i=1}^{p} \phi_i X(n-i) + \sum_{i=1}^{q} \theta_i Z(n-i) + Z(n)$$

  where $Z(n)$ is a white Gaussian process.

- ***Filter implementation:***



- ***Causal and stable ARMA processes:***
  An ARMA(p, q) process $X(n)$ is called causal and stable if there exists an infinite causal and stable transversal filter with impulse response $h(n)$ such that

$$X(n) = \sum_{i=0}^{\infty} h(i)Z(n-i) = h(n)*Z(n)$$

  Let us define

$$\theta(z) \equiv 1 + \sum_{i=1}^{q} \theta_i z^{-i} \qquad \text{and} \qquad \phi(z) \equiv 1 - \sum_{i=1}^{p} \phi_i z^{-i}$$

  A necessary and sufficient condition for an ARMA(p, q) process to be causal and stable is that the polynomial $\phi(z)$ has its roots inside the unit circle.
  The impulse response of a causal and stable ARMA(p, q) process is then determined by the identity

$$\sum_{i=0}^{\infty} h_i z^{-i} = \frac{\theta(z)}{\phi(z)} \qquad |z| \geq 1$$

  In the above considerations we assume that $\theta(z)$ and $\phi(z)$ have no common root.

- ***Transfer function of the filter:***

$$H(f) = \frac{1 + \sum_{i=1}^{q} \theta_i \exp(-j2\pi i f)}{1 - \sum_{i=1}^{p} \phi_i \exp(-j2\pi i f)}$$

  Proof: Similar as before.

- ***Power spectrum of an ARMA(p,q) process:***

$$S_{XX}(f) = \frac{\left|1 + \sum_{i=1}^{q} \theta_i \exp(-j2\pi i f)\right|^2}{\left|1 - \sum_{i=1}^{p} \phi_i \exp(-j2\pi i f)\right|^2} \sigma_Z^{\,2}$$

- ***Mean value and autocorrelation function of a causal ARMA(p,q) process:***
  If the ARMA process $X(n)$ is causal,

$$\mu_X = 0$$
$$R_{XX}(k) = \sigma_Z^2 R_{hh}(k)$$

- ***Importance of ARMA(p,q) processes:***
  - Because of the linearity property of ARMA(p,q) processes, analytical expressions can be derived which describe their statistical behavior, i.e. their autocorrelation and power spectrum.
  - For any given zero-mean WSS process $Y(n)$ with autocorrelation function $R_{YY}(k)$ there exists an ARMA(p,q) process $X(n)$ such that

$$R_{YY}(k) = R_{XX}(k) \qquad |k| \leq K.$$

  In this sense, any WSS process can be approximated by an ARMA(p,q) process.