1. Signal Detection

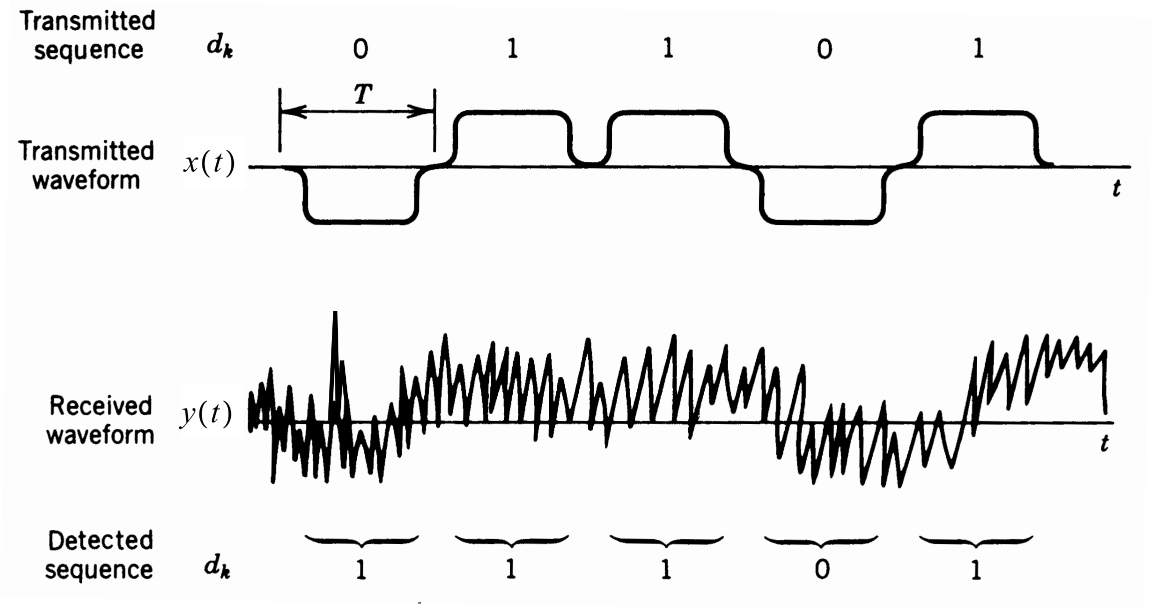
1.1. Hypothesis testing

- ***Method:***
  In hypothesis testing, a decision is made based on the observation of a random variable as to which of several hypotheses to accept.
  In binary hypothesis testing the choice is made among two hypotheses.

  Example 1: *Detection of a BPAM signal:*

  H_0: "0" is transmitted Null hypothesis

  H_1: "1" is transmitted Alternate hypothesis



  Example 2: *Target detection in radar technique:*

  H_0: The target is not present

  H_1: The target is present

- ***Mathematical framework for binary hypothesis testing:***

  - **Two hypotheses:** H_0 and H_1

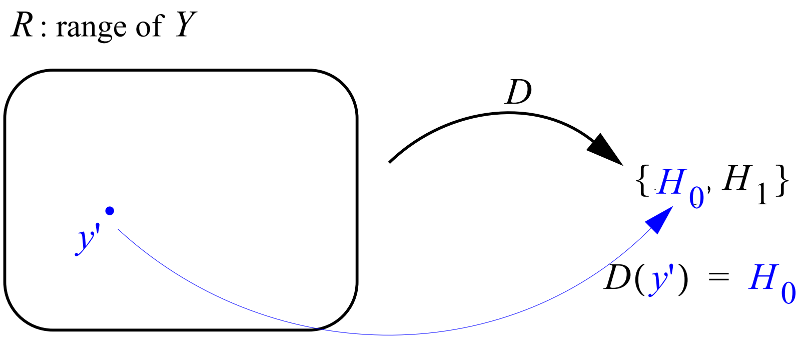
  - **One observation** y **of a random variable** Y **whose probability density function (pdf) under each hypothesis is known.**
    We denote these pdfs as:

$$f(y|H_0) \text{ and } f(y|H_1)$$

- **A decision rule** D, i.e. a mapping

$$D : R \to \{H_0, H_1\},\ y \to D(y),$$

where R denotes the range of Y.

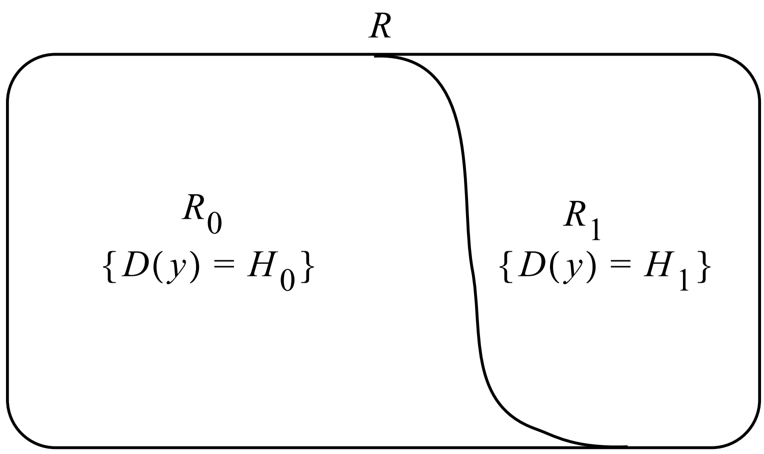


In the above figure, when $Y = y'$ is observed, the decision $D(y') = H_0$ is made.

The decision rule D determines two **decision regions** in R:

$$R_0 : y \in R_0 \quad \Rightarrow \quad D(y) = H_0$$

$$R_1 : y \in R_1 \quad \Rightarrow \quad D(y) = H_1$$



Properties of R_0 and R_1:

- $R_0 \cup R_1 = R$
- $R_0 \cap R_1 = \varnothing$

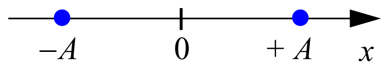
Example: Binary pulse amplitude modulation (BPAM)

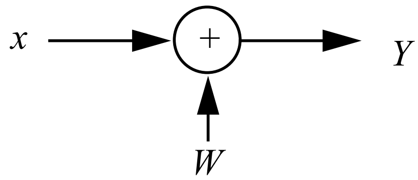
- Signal model:

$$Y = x + W$$

where

$$x = \begin{cases} -A & \text{under} \quad H_0 \\ +A & \text{under} \quad H_1 \end{cases}$$





W is a Gaussian noise, i.e.:

- W is a Gaussian random variable,
- with expectation $\mathbf{E}[W] = 0$,
- and variance $\mathbf{E}[W^2] = \sigma^2$.

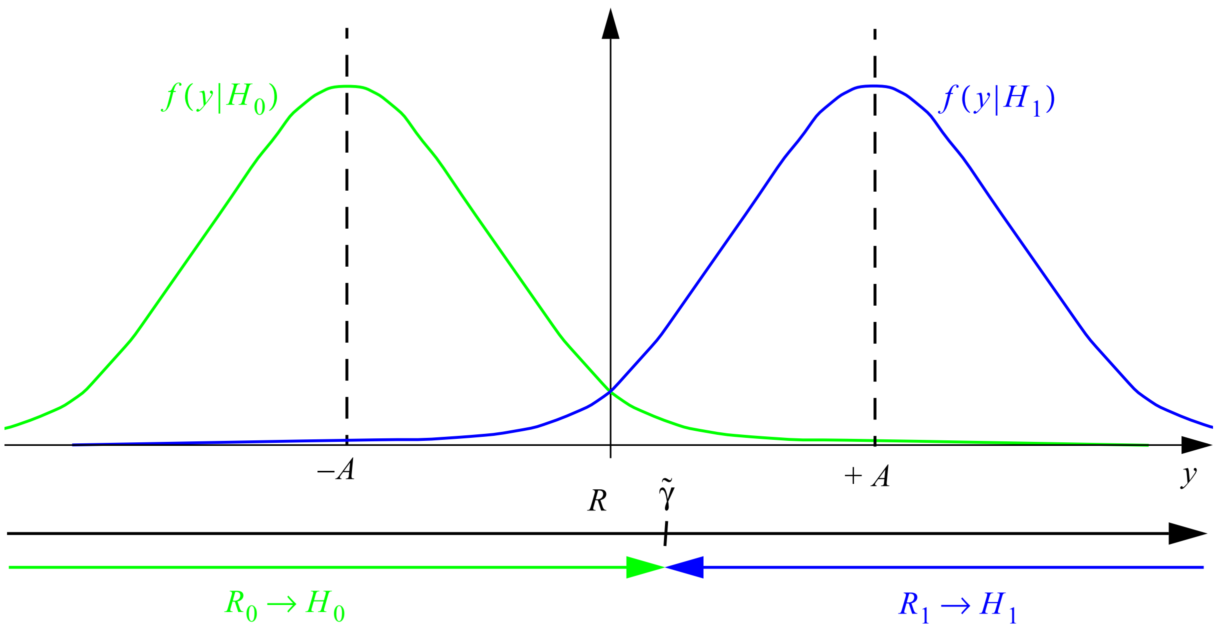
- Probability density function (pdf) of W:

$$f(w) = \frac{1}{\sqrt{2\pi}\sigma}\exp\left\{-\frac{1}{2\sigma^2}w^2\right\}$$

- Pdf of Y under H_0 and H_1 and decision regions:

$$f(y|H_0) = f(w)|_{w = y+A} = \frac{1}{\sqrt{2\pi}\sigma}\exp\left\{-\frac{1}{2\sigma^2}(y+A)^2\right\}$$

$$f(y|H_1) = f(w)|_{w = y-A} = \frac{1}{\sqrt{2\pi}\sigma}\exp\left\{-\frac{1}{2\sigma^2}(y-A)^2\right\}$$



- ***Decision table:***

Decision D	True hypothesis H	
	H_0	H_1
H_0	(H_0, H_0)	(H_0, H_1)
H_1	(H_1, H_0)	(H_1, H_1)

- (green) : Correct decision
- (red) : Incorrect decision:

The pair (H_i, H_j) $(i, j = 0, 1)$ in the above table means $D = H_i$ and $H = H_j$.

- ***Probabilities of correct decision and of making an error:***

- Probability of correct decision:

$$\begin{aligned} P_c &= P[D = H_0, H = H_0] + P[D = H_1, H = H_1] \\ &= P[D = H_0 | H_0]P[H_0] + P[D = H_1 | H_1]P[H_1] \end{aligned}$$

- Probability of incorrect decision:

$$\begin{aligned} P_e &= P[D = H_1, H = H_0] + P[D = H_0, H = H_1] \\ &= P[D = H_1 | H_0]P[H_0] + P[D = H_0 | H_1]P[H_1] \end{aligned}$$

Obviously,

$$P_c = 1 - P_e.$$

- ***Types of error and their probability:***

- **False alarm** (Type I error): $D = H_1$ when H_0 is true.

False alarm probability:

$$P_f \equiv P[D = H_1 | H_0].$$

- **Miss** (Type II error): $D = H_0$ when H_1 is true.

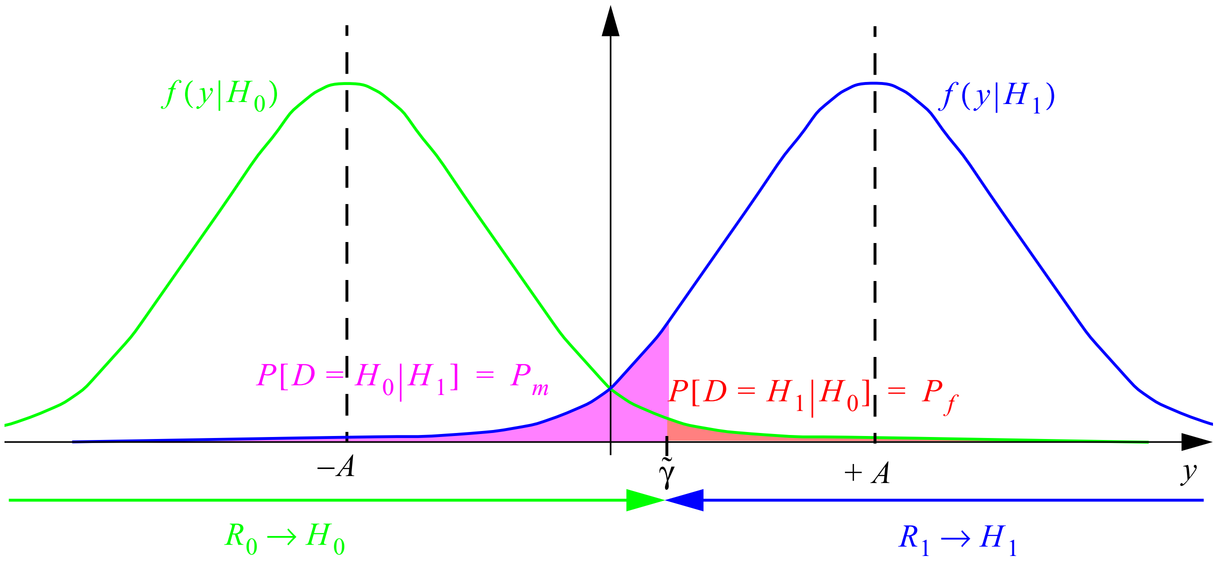
Probability of a miss:

$$P_m \equiv P[D = H_0 | H_1].$$

- Probability of incorrect decision:

$$P_e = P_f P[H_0] + P_m P[H_1]$$

Example: BPAM (cont'd):



1.2. Decision rules

1.2.1. Maximum "a posteriori" (MAP) decision rule

We seek a decision rule which minimizes the probability of error P_e.

- ***MAP decision rule:***
  Such a rule exists. It is of the form:

$$P[H_1|y] \underset{H_0}{\overset{H_1}{\gtrless}} P[H_0|y]$$

where $P[H_i|y]$, $i = 0, 1$, is the "a posteriori" probability of H_i when $Y = y$ is observed.

- ***Bayes rule:***
  The "a posteriori" probability $P[H_i|y]$ can be obtained by invoking Bayes' rule:

$$P[H_i|y] = \frac{f(y|H_i)P[H_i]}{f(y)}.$$

- ***MAP decision rule (cont'd):***
  Using the last identity, the MAP decision rule can be recast as:

$$f(y|H_1)P[H_1] \underset{H_0}{\overset{H_1}{\gtrless}} f(y|H_0)P[H_0]$$

or equivalently

$$L(y) \equiv \frac{f(y|H_1)}{f(y|H_0)} \underset{H_0}{\overset{H_1}{\gtrless}} \frac{P[H_0]}{P[H_1]}$$

The function $L(y)$ is called the likelihood ratio.
It is more common to use the log-likelihood ratio

$$l(y) \equiv \ln(L(y))$$

instead:

$$l(y) = \ln\left(\frac{f(y|H_1)}{f(y|H_0)}\right) \underset{H_0}{\overset{H_1}{\gtrless}} \ln\left(\frac{P[H_0]}{P[H_1]}\right)$$

- ***Derivation of the MAP decision rule:***

□

Example: BPAM (cont'd)

- Likelihood ratio:

$$L(y) = \frac{f(y|H_1)}{f(y|H_0)}$$

$$= \frac{\exp\left\{-\frac{1}{2\sigma^2}(y-A)^2\right\}}{\exp\left\{-\frac{1}{2\sigma^2}(y+A)^2\right\}} = \exp\left\{\frac{1}{2\sigma^2}[(y+A)^2-(y-A)^2]\right\}$$

- Loglikelihood ratio:

$$l(y) = \ln\left(\frac{f(y|H_1)}{f(y|H_0)}\right)$$

$$= \frac{1}{2\sigma^2}[(y+A)^2-(y-A)^2]$$

$$= \frac{2Ay}{\sigma^2}$$

- MAP decision rule:

$$\frac{2Ay}{\sigma^2} \underset{H_0}{\overset{H_1}{\gtrless}} \ln\left(\frac{P[H_0]}{P[H_1]}\right)$$

$$y \underset{H_0}{\overset{H_1}{\gtrless}} \frac{\sigma^2}{2A}\ln\left(\frac{P[H_0]}{P[H_1]}\right) \equiv \tilde{\gamma}_{MAP}$$

- ***Maximum likelihood (ML) decision rule:***
  Selecting a uniform "a priori" pdf for the hypotheses, i.e.

$$P[H_0] = P[H_1] = \frac{1}{2},$$

the MAP decision rule reduces to the ML decision rule:

$$f(y|H_1) \underset{H_0}{\overset{H_1}{\gtrless}} f(y|H_0)$$

The ML decision rule selects the hypothesis which maximizes the likelihood function $H \to f(y|H)$.

1.2.2. Bayes decision rule

- ***Cost function:***
  In many engineering branches costs have to be taken into account depending on the decision and the true hypothesis.

Decision D	True hypothesis	
	H_0	H_1
H_0	C_{00}	C_{01}
H_1	C_{10}	C_{11}

Usually, the cost of making a wrong decision is higher than that of making a correct decision:

$$C_{10} \geq C_{00} \qquad \text{and} \qquad C_{01} \geq C_{11}.$$

- ***Average cost:***

$$\begin{aligned}\bar{C} \;=\; & C_{00}P[D = H_0|H_0]P[H_0] + C_{10}P[D = H_1|H_0]P[H_0] + \\ & + C_{01}P[D = H_0|H_1]P[H_1] + C_{11}P[D = H_1|H_1]P[H_1]\end{aligned}$$

- ***Bayes decision rule:***

A Bayes decision rule is a decision rule which minimizes the average cost $\bar{C}$. It is of the form:

$$L(y) \;=\; \frac{f(y|H_1)}{f(y|H_0)} \underset{H_0}{\overset{H_1}{\gtrless}} \frac{P[H_0](C_{10} - C_{00})}{P[H_1](C_{01} - C_{11})}$$

Proof:

The average cost $\bar{C}$ can be written as:

$$\begin{aligned}\bar{C} \;=\; & C_{10}P[H_0] + C_{11}P[H_1] + \\ & + \int_{R_0} \{P[H_1](C_{01} - C_{11})f(y|H_1) - P[H_0](C_{10} - C_{00})f(y|H_0)\}dy\end{aligned}$$

□

Note that the Bayes rule reduces to the MAP rule when the cost is selected to be:

$$C_{00} \;=\; C_{11} \;=\; 0$$
$$C_{10} \;=\; C_{01} \;=\; 1$$

1.2.3. Minimax and Neyman-Pearson decision rule

Bayes decision rules necessitate the specification of an “a priori” pdf:

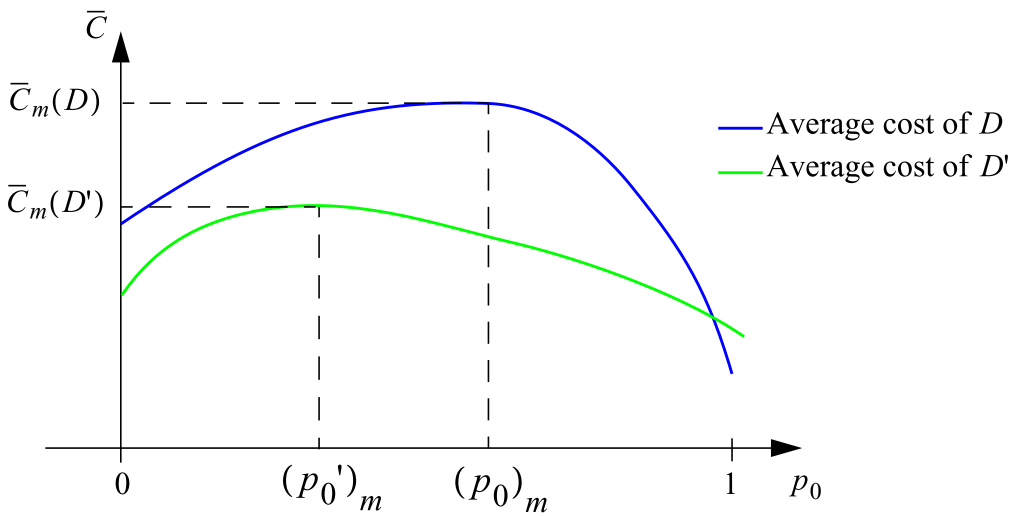
$$P[H_0],\ P[H_1] = 1 - P[H_0].$$

In some situations such a pdf is unknown and difficult to assess or even the definition of such a pdf does not make sense. In this case we have to resort to alternative decision rules.

- ***Minimax decision rule:***
  The minimax decision rule is employed when the “a priori” pdf is unknown.

  - **Maximum average cost of a decision rule:**

    Let us consider the behavior of $\bar{C}$ for a fixed decision rule D as $p_0 \equiv P[H_0]$ varies:



    $\bar{C}_m(D)$ is the maximum average cost and

    $P[H_0] = (p_0)_m$, $P[H_1] = 1-(p_0)_m$ is the worst case “a priori” pdf

    when employing decision rule D.

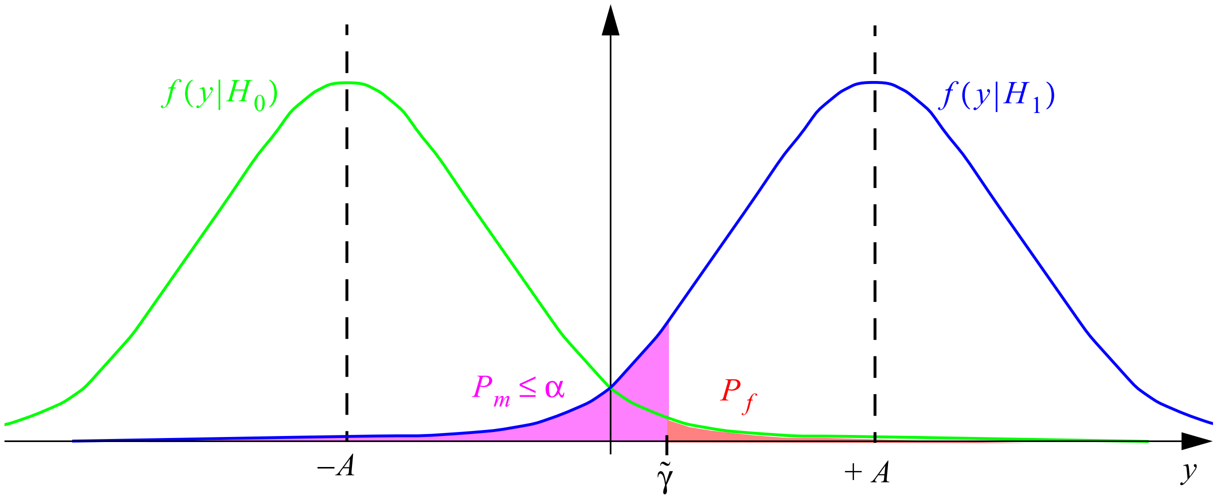
  - **Minimax decision rule:**

    A minimax decision rule, say D_{MM} minimizes $\bar{C}_m$:

$$\bar{C}_m(D_{MM}) \le \bar{C}_m(D) \qquad \text{for any decision rule } D$$

Minimax $\equiv$ **mini**mize the **max**imum of $\bar{C}_m$.

- ***Neyman-Pearson decision rule:***
  The Neyman-Pearson (NP) decision rule is used when neither an "a priori" pdf nor cost assignments are given.
  A NP decision rule minimizes the probability of false alarm $P_f = P[D = H_1 | H_0]$ while keeping the probability of a miss $P_m = P[D = H_0 | H_1]$ below a certain specified level, say α.
  Example: BPAM (cont'd)



Thus, a NP decision rule D_{NP_α} satisfies the inequality

$$P_f(D_{NP_\alpha}) \leq P_f(D) \quad \text{for any decision rule } D \text{ such that } P_m(D) \leq \alpha$$

1.2.4. General form of a binary decision rule:

Both minimax and NP decision rules can be shown to be of the form:

$$L(y) = \frac{f(y|H_1)}{f(y|H_0)} \underset{H_0}{\overset{H_1}{\gtrless}} \gamma$$

or equivalently

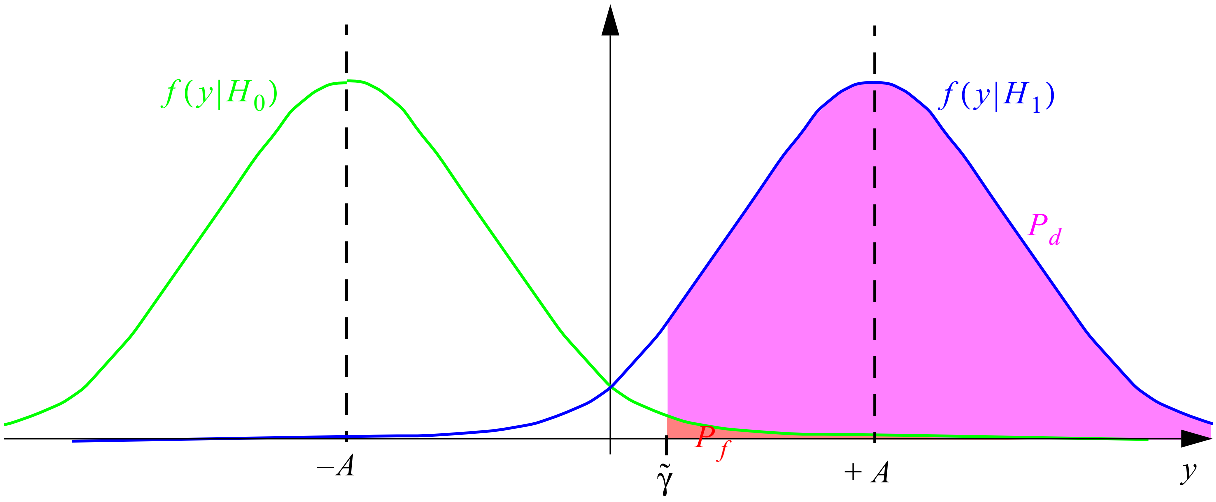
$$l(y) = \ln\left(\frac{f(y|H_1)}{f(y|H_0)}\right) \underset{H_0}{\overset{H_1}{\gtrless}} \ln(\gamma)$$

for some decision threshold γ.
Notice that the Bayes decision rule, and therefore the MAP decision rule as well, are also of the same form with

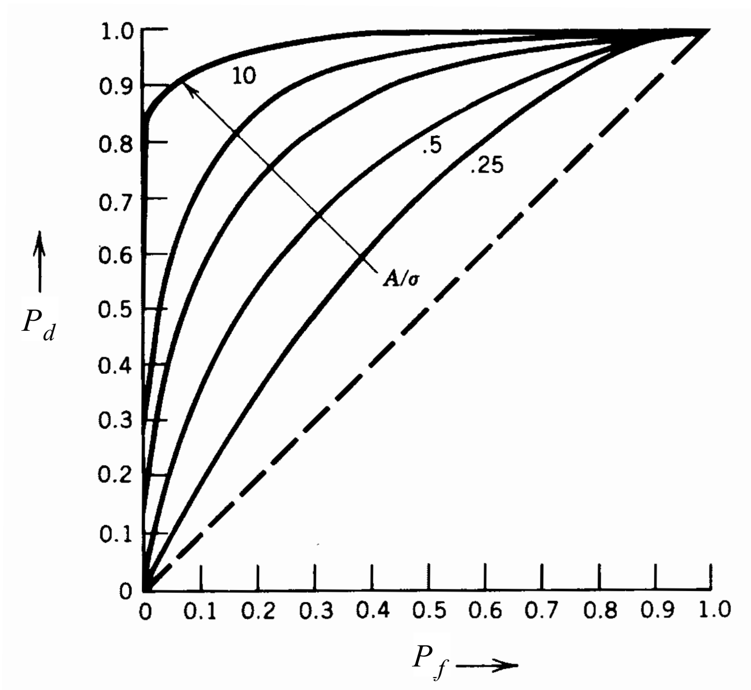
$$\gamma = \frac{P[H_0](C_{10} - C_{00})}{P[H_1](C_{01} - C_{11})}.$$

- ***Receiver operating characteristics (ROC):***
  In radar technique the performance of detectors are described in terms of a graph representing the probability of correct detection $P_d = P[D = H_1|H_1]$ versus the false alarm probability $P_f = P[D = H_1|H_0]$:

  Example: BPAM (cont'd)



ROC for $\gamma = \gamma_{MAP} = 1 \Rightarrow \tilde{\gamma} = \tilde{\gamma}_{MAP} = 0$:



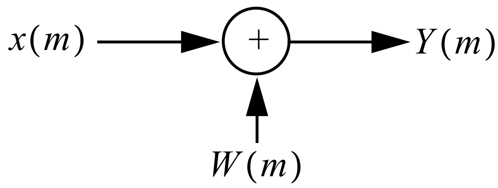
1.3. Binary detection of discrete-time signals

1.3.1. Time-limited discrete-time signals

- ***Signal model:***

$$Y(m) = x(m) + W(m)$$

$$m = 0, \ldots, M-1$$

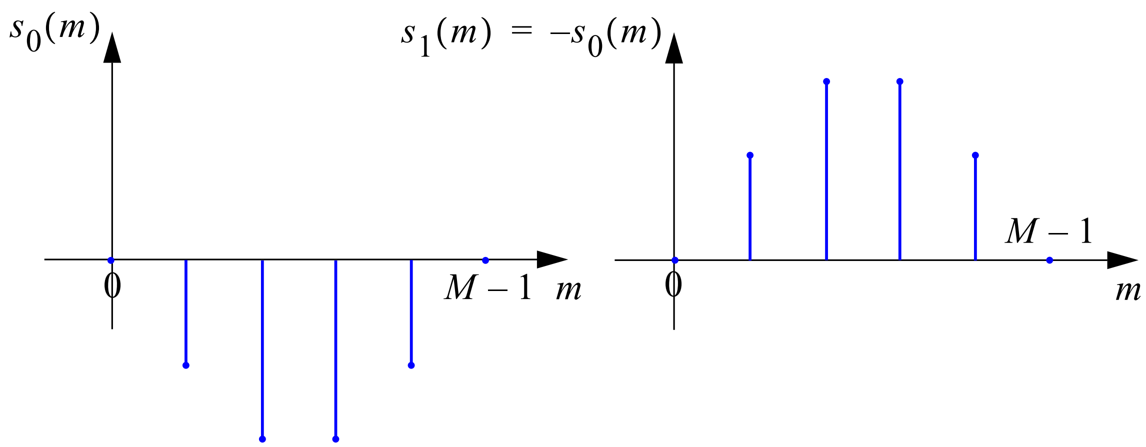


where

- $x(m) = \begin{cases} s_0(m) & \text{under} \quad H_0 \\ s_1(m) & \text{under} \quad H_1 \end{cases}$

- $W(m)$ is a white Gaussian noise:
    - $W(m)$ is a Gaussian process,
    - $\mathbf{E}[W(m)] = 0$,
    - $\mathbf{E}[W(m)W(m+k)] = \sigma^2\delta(k)$.

Example: *Detection of BPAM signals*



- ***Vector representation of finite sequences:***
    - Deterministic signals:

$$\boldsymbol{u} \equiv [u(0), \ldots, u(M-1)]^T$$

- Random sequences:

$$\boldsymbol{U} \equiv [U(0), \ldots, U(M-1)]^T$$

- ***Pdf of Y under H_0 and H_1:***

  - Vector representation of the received signal:

$$\boldsymbol{Y} = \boldsymbol{x} + \boldsymbol{W}$$

  where

$$\boldsymbol{x} = \begin{cases} \boldsymbol{s}_0 & \text{under} \quad H_0 \\ \boldsymbol{s}_1 & \text{under} \quad H_1 \end{cases}$$

  - Pfd of $\boldsymbol{W}$:

$$f(\boldsymbol{w}) = \prod_{m=0}^{M-1} f(w(m))$$

$$= \prod_{m=0}^{M-1} \frac{1}{\sqrt{2\pi}\sigma} \exp\left\{-\frac{1}{2\sigma^2} w(m)^2\right\}$$

$$= \left(\frac{1}{\sqrt{2\pi}\sigma}\right)^M \exp\left\{-\frac{1}{2\sigma^2}\|\boldsymbol{w}\|^2\right\}$$

  where

$$\|\boldsymbol{w}\| \equiv \sqrt{\sum_{m=0}^{M-1} w(m)^2}$$

  is the norm of $\boldsymbol{w}$.

  - Pdf of $\boldsymbol{Y}$ under H_0 and H_1:

$$f(\boldsymbol{y}|H_i) = f(\boldsymbol{w})|_{\boldsymbol{w} = \boldsymbol{y} - \boldsymbol{s}_i}$$

$$= \left(\frac{1}{\sqrt{2\pi}\sigma}\right)^M \exp\left\{-\frac{1}{2\sigma^2}\|\boldsymbol{y} - \boldsymbol{s}_i\|^2\right\}$$

- ***Likelihood and loglikelihood ratios:***
  - Likelihood ratio:

$$L(\boldsymbol{y}) = \frac{f(\boldsymbol{y}|H_1)}{f(\boldsymbol{y}|H_0)} = \frac{\exp\left\{-\frac{1}{2\sigma^2}\|\boldsymbol{y}-\boldsymbol{s}_1\|^2\right\}}{\exp\left\{-\frac{1}{2\sigma^2}\|\boldsymbol{y}-\boldsymbol{s}_0\|^2\right\}} = \exp\left\{\frac{1}{2\sigma^2}[\|\boldsymbol{y}-\boldsymbol{s}_0\|^2 - \|\boldsymbol{y}-\boldsymbol{s}_1\|^2]\right\}$$

  - Loglikelihood ratio:

$$l(\boldsymbol{y}) = \ln\left(\frac{f(\boldsymbol{y}|H_1)}{f(\boldsymbol{y}|H_0)}\right) = \frac{1}{2\sigma^2}[\|\boldsymbol{y}-\boldsymbol{s}_0\|^2 - \|\boldsymbol{y}-\boldsymbol{s}_1\|^2] = \frac{1}{\sigma^2}\left[\boldsymbol{y}^T(\boldsymbol{s}_1-\boldsymbol{s}_0) + \frac{1}{2}(\|\boldsymbol{s}_0\|^2 - \|\boldsymbol{s}_1\|^2)\right]$$

- ***Decision rules:***

$$l(\boldsymbol{y}) = \frac{1}{\sigma^2}\left[\boldsymbol{y}^T(\boldsymbol{s}_1-\boldsymbol{s}_0) + \frac{1}{2}(\|\boldsymbol{s}_0\|^2 - \|\boldsymbol{s}_1\|^2)\right] \underset{H_0}{\overset{H_1}{\gtrless}} \ln(\gamma)$$

or equivalently

$$\boldsymbol{y}^T(\boldsymbol{s}_1-\boldsymbol{s}_0) \underset{H_0}{\overset{H_1}{\gtrless}} \sigma^2\ln(\gamma) + \frac{1}{2}(E_{\boldsymbol{s}_1} - E_{\boldsymbol{s}_0})$$

where

$$E_{\boldsymbol{s}_i} \equiv \|\boldsymbol{s}_i\|^2 = \sum_{m=0}^{M-1} s_i(m)^2$$

is the energy of the signal $\boldsymbol{s}_i$, $i = 0, 1$.

Comment:

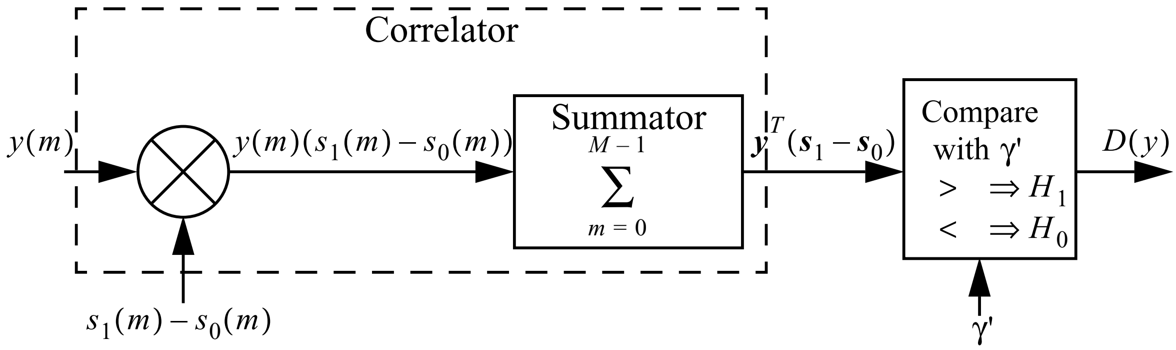
Notice that $\boldsymbol{y}^T(\boldsymbol{s}_1 - \boldsymbol{s}_0)$ is the scalar product of $\boldsymbol{y}$ and $\boldsymbol{s}_1 - \boldsymbol{s}_0$, i.e.

$$\boldsymbol{y}^T(\boldsymbol{s}_1 - \boldsymbol{s}_0) = \sum_{m=0}^{M-1} y(m)(s_1(m) - s_0(m))$$

Special case: MAP decision rule:

$$\boldsymbol{y}^T(\boldsymbol{s}_1 - \boldsymbol{s}_0) \underset{H_0}{\overset{H_1}{\gtrless}} \sigma^2 \ln\left(\frac{P[H_0]}{P[H_1]}\right) + \frac{1}{2}(E_{\boldsymbol{s}_1} - E_{\boldsymbol{s}_0})$$

- ***Block diagram of a binary detector for time-limited discrete-time signals:***



$$\gamma' = \sigma^2 \ln(\gamma) + \frac{1}{2}(E_{s_1} - E_{s_0})$$

1.3.2. Discrete-time signals with finite energy

- ***Signal model:***

$$Y(m) = x(m) + W(m) \qquad m = \ldots, -2, -1, 0, 1, 2, \ldots$$

where

$$\text{-}\ x(m) = \begin{cases} s_0(m) & \text{under} \quad H_0 \\ s_1(m) & \text{under} \quad H_1 \end{cases}$$

where $s_i(m)$, $i = 0, 1$ has finite energy, i.e.

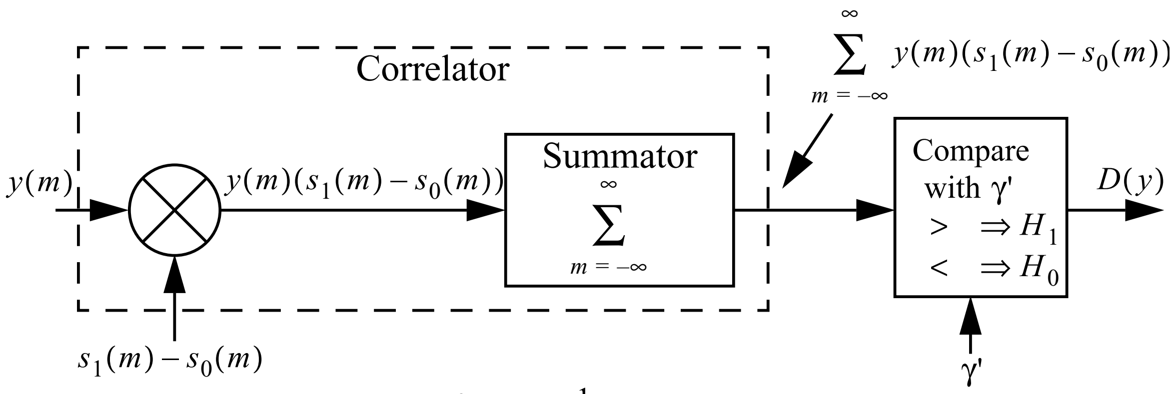
$$E_{s_i} \equiv \sum_{m=-\infty}^{\infty} s_i(m)^2 < \infty$$

- $W(m)$ is a white Gaussian noise sequence with variance σ^2.

- ***Decision rules:***
  Decision rules can be obtained in this case as well which prove to be of the form:

$$\sum_{m=-\infty}^{\infty} y(m)(s_1(m) - s_0(m)) \underset{H_0}{\overset{H_1}{\gtrless}} \sigma^2 \ln(\gamma) + \frac{1}{2}(E_{s_1} - E_{s_0})$$

Hence, these decision rules are essentially the same as those derived for time-limited sequences.



$$\gamma' = \sigma^2 \ln(\gamma) + \frac{1}{2}(E_{s_1} - E_{s_0})$$

1.4. Binary detection of continuous-time signals

- ***Signal model:***

with $$Y(t) = x(t) + W(t)$$

- $x(t) = \begin{cases} s_0(t) & \text{under} \quad H_0 \\ s_1(t) & \text{under} \quad H_1 \end{cases}$.

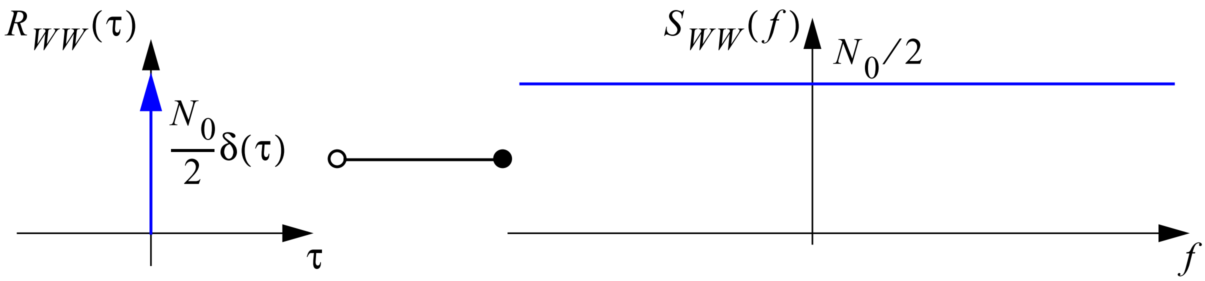
  The signals $s_0(t)$ and $s_1(t)$ have finite energy, i.e.

$$E_{s_i} \equiv \int_{-\infty}^{\infty} s_i(t)^2 dt < \infty \qquad i = 0, 1$$

- $W(t)$ is a white Gaussian noise:
  - $W(t)$ is a Gaussian process,
  - $\mathbf{E}[W(t)] = 0$,
  - $R_{WW}(\tau) = \mathbf{E}[W(t)W(t+\tau)] = \frac{N_0}{2}\delta(\tau)$.

The power spectral density function of $W(t)$ reads:

$$S_{WW}(f) = F\{R_{WW}(\tau)\} = \frac{N_0}{2}$$

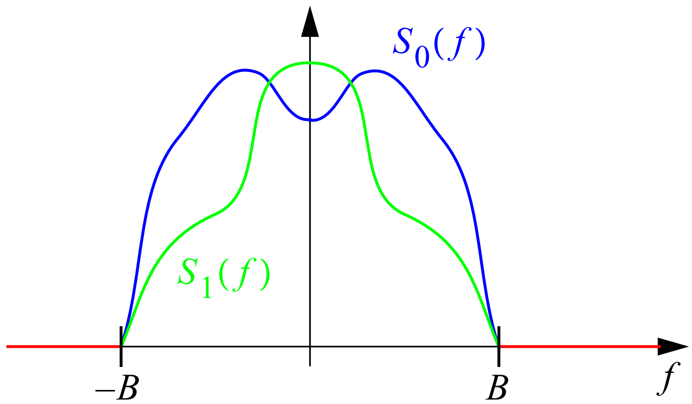


- ***Key issue:***
  Subsequently we consider two situations which prove to be equivalent to that previously considered in Sect. 3.3.2.

1.4.1. Bandwidth-limited continuous-time signals

- ***Bandwidth-limited signals:***
  The signals $s_0(t)$ and $s_1(t)$ are bandwidth-limited with bandwidth B:

Spectrum of $s_0(t)$ and $s_1(t)$:



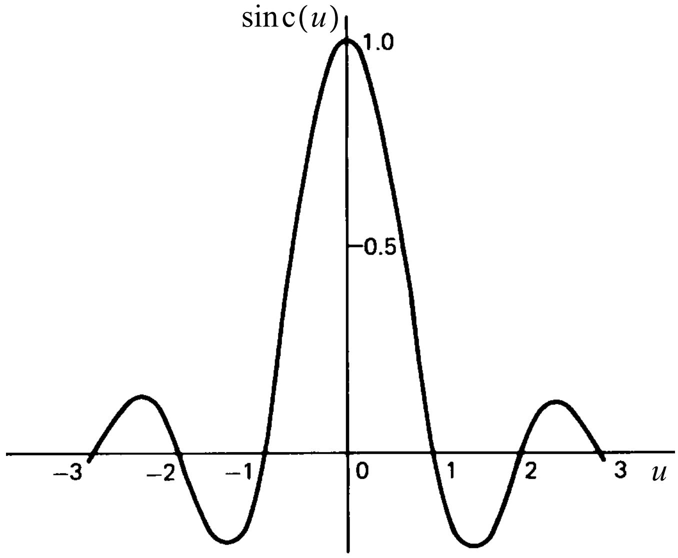
- ***Sampling theorem for bandwidth-limited signals:***
  The signal $s_i(t)$, $i = 0, 1$, can be represented as

$$s_i(t) = \sum_{m=-\infty}^{\infty} s_i(mT_s)\,\mathrm{sinc}\left(\frac{1}{T_s}(t - mT_s)\right),$$

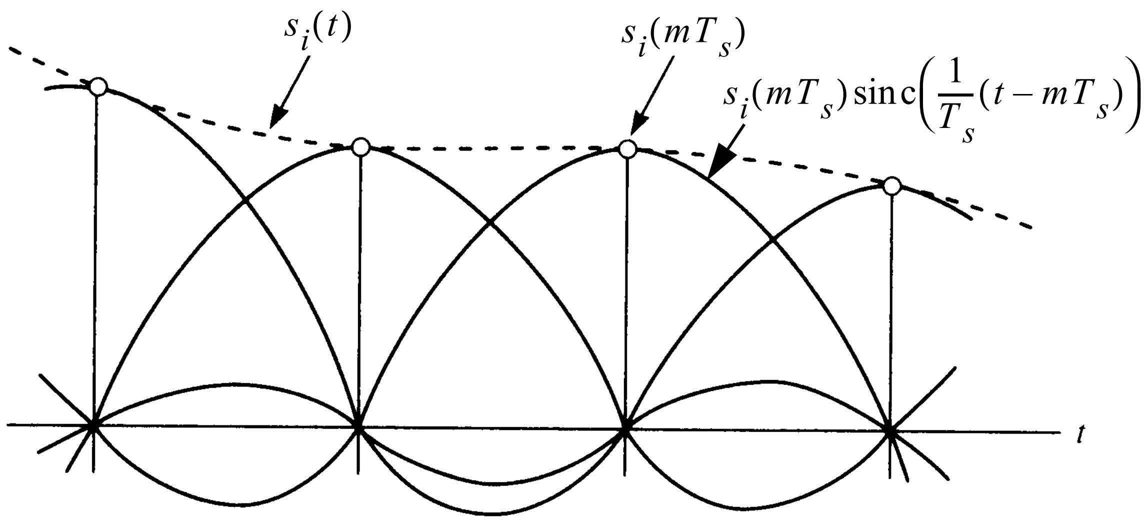
where

- $\frac{1}{T_s} = 2B$ is the Nyquist rate

- $\mathrm{sinc}(u) \equiv \frac{\sin(\pi u)}{\pi u}$:



According to the above sum, $s_i(t)$ is entirely determined by its samples $s_i(mT_s)$, $m = \ldots, -1, 0, 1, \ldots$:



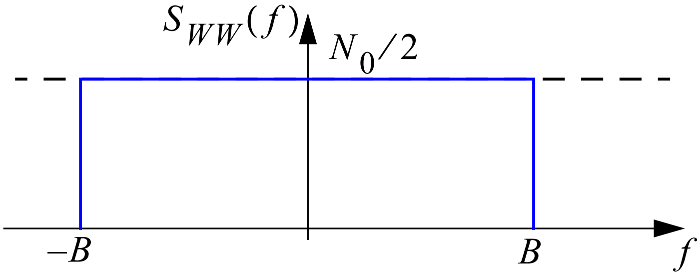
- ***Parseval relationship for bandwidth-limited finite-energy signals:***
  Let $u(t)$ and $v(t)$ denote two finite-energy signals with bandwidth B. Then

$$\int_{-\infty}^{\infty} u(t)v(t)dt = T_s \sum_{m=-\infty}^{\infty} u(mT_s)v(mT_s) .$$

In particular,

$$E_u = \int_{-\infty}^{\infty} u(t)^2 dt = T_s \sum_{m=-\infty}^{\infty} u(mT_s)^2 .$$

- ***Sampling theorem for bandwidth-limited processes:***
  Without loss of generality we can assume that $W(t)$ is bandwidth-limited with bandwidth B.



Then,

$$W(t) = \sum_{m=-\infty}^{\infty} W(mT_s)\operatorname{sinc}\left(\frac{1}{T_s}(t - mT_s)\right),$$

where $W(mT_s)$ is the sample of $W(t)$ at $t = mT_s$.

Moreover, it can be shown that the sequence $W(mT_s)$, $m = \ldots, -1, 0, 1, \ldots$, is a white Gaussian sequence with variance

$$\mathbf{E}[W(mT_s)^2] = \int S_{WW}(f)df = N_0 B .$$

- ***Decision rules:***
  The identification

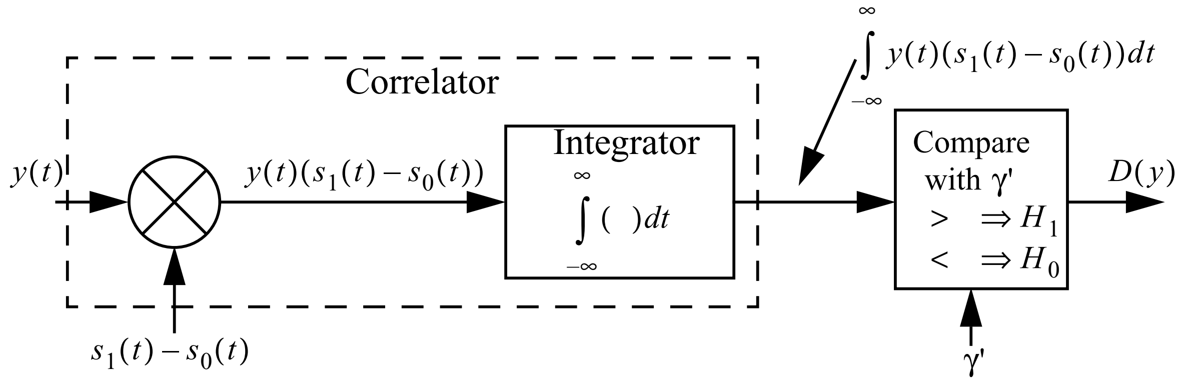
$$s_i(t) \quad \leftrightarrow \quad s_i(m) = s_i(mT_s),\ m = \ldots, -1, 0, 1, \ldots$$

$$W(t) \quad \leftrightarrow \quad W_i(m) = W_i(mT_s),\ m = \ldots, -1, 0, 1, \ldots$$

in combination with the Parseval relation allows to reduce the situation of continuous-time bandlimited signals with finite energy to the case of infinite sequences with finite energy considered in Sect. 3.3.2.
Invoking the Parseval relation again and the result obtained in Sect. 3.3.2, the decision rules are found to be of the form:

$$\int_{-\infty}^{\infty} y(t)(s_1(t) - s_0(t))dt \underset{H_0}{\overset{H_1}{\gtrless}} \frac{N_0}{2}\ln(\gamma) + \frac{1}{2}(E_{s_1} - E_{s_0})$$

- ***Block diagram of a binary detector for bandwidth-limited continuous-time signals:***



$$\gamma' = \sigma^2 \ln(\gamma) + \frac{1}{2}(E_{s_1} - E_{s_0})$$

1.4.2. Time-limited continuous-time signals

A similar rational as used above can be applied to time-limited (but possibly bandwidth unlimited) finite-energy signals.

- ***Time-limited signals:***

Let $[T_u, T_o]$ be an interval outside which $s_0(t)$ and $s_1(t)$ vanish, i.e.

$$s_i(t) = 0 \qquad t \notin [T_u, T_o] \qquad i = 0, 1 .$$

- ***Decision rules:***

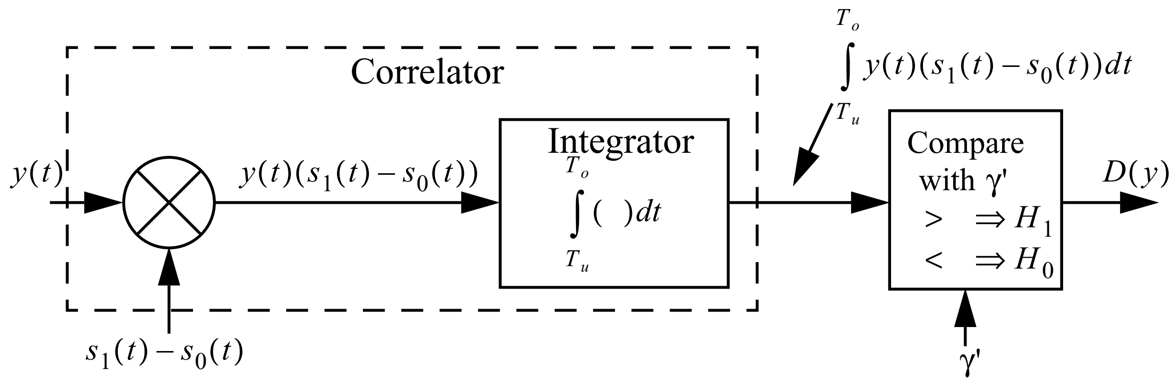
The decision rules are found to be of the form:

$$\int_{T_u}^{T_o} y(t)(s_1(t)-s_0(t))dt \underset{H_0}{\overset{H_1}{\gtrless}} \frac{N_0}{2}\ln(\gamma) + \frac{1}{2}(E_{s_1} - E_{s_0})$$

where

$$E_{s_i} = \int_{T_u}^{T_o} s_i(t)^2 dt$$

- ***Block diagram of a binary detector for time-limited continuous-time signals:***

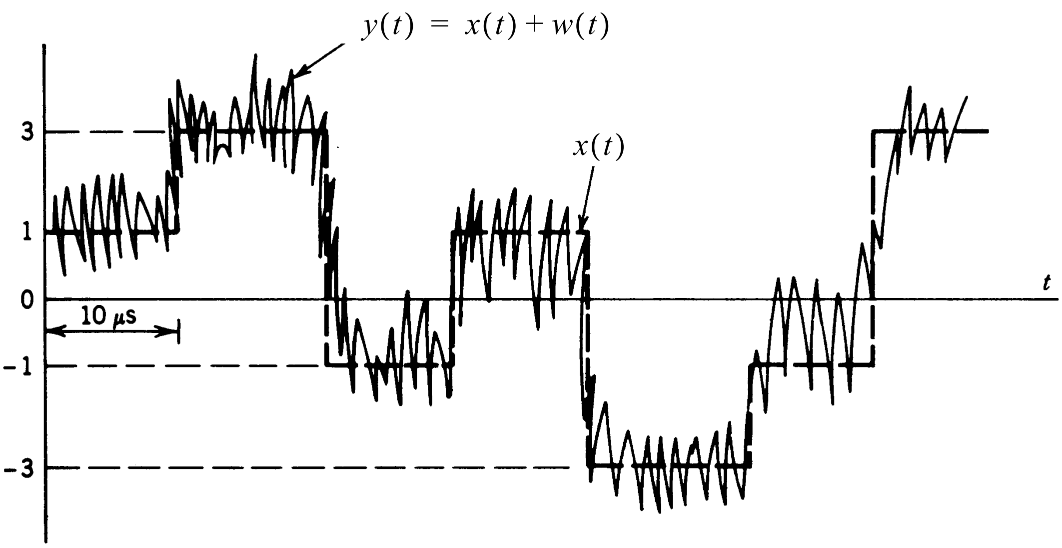


$$\gamma' = \sigma^2 \ln(\gamma) + \frac{1}{2}(E_{s_1} - E_{s_0})$$

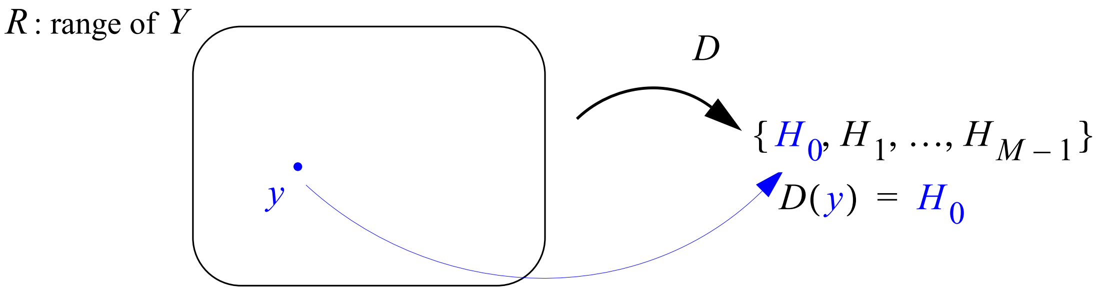
1.5. M-ary detection

- ***Multiple hypothesis testing:***
  So far, we have considered the problem of deciding between one among two hypotheses. In many engineering problems a decision must be taken between more that two possibilities, say $H_0, \ldots, H_{M-1}$.

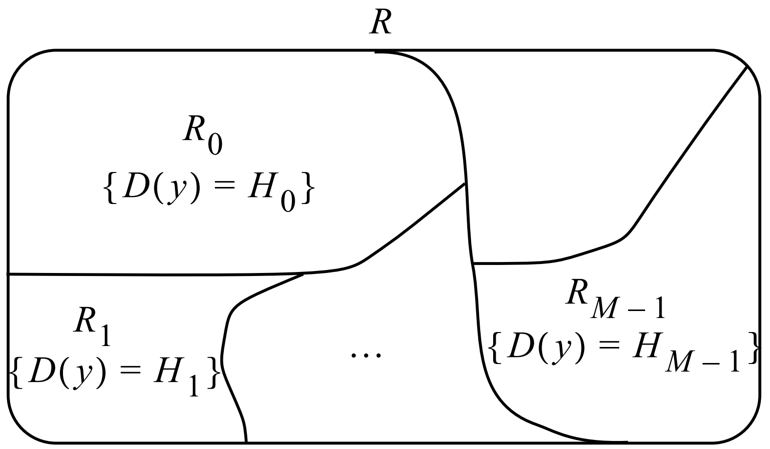
  Example: 4-PAM



- ***Decision rule and decision regions:***
  - Decision rule:



  - Decision regions:



$$R_0 \cup R_1 \cup \ldots \cup R_{M-1} = R$$
$$R_0 \cap R_1 \cap \ldots \cap R_{M-1} = \varnothing$$

- ***Probabilities of correct decision and of making an error:***
  - Probability of correct decision:

$$P_c = \sum_{i=0}^{M-1} P[D = H_i, H = H_i] = \sum_{i=0}^{M-1} P[D = H_i | H_i] P[H_i]$$

  - Probability of incorrect decision:

$$P_e = 1 - P_c$$

- ***MAP for M-ary detection:***
  In order to minimize P_e a decision rule must select an hypothesis whose "a posteriori" probability is maximum:

$$\text{Select } H_i \text{ if } P[H_i|y] \geq P[H_j|y] \text{ for any } j = 0, \ldots, M-1.$$

or equivalently, by invoking Bayes' rule:

$$\text{Select } H_i \text{ if } \frac{f(y|H_i)}{f(y|H_j)} \geq \frac{P[H_j]}{P[H_i]} \text{ for any } j = 0, \ldots, M-1.$$

Using the log-likelihood function:

$$\text{Select } H_i \text{ if } \ln\left(\frac{f(y|H_i)}{f(y|H_j)}\right) \geq \ln\left(\frac{P[H_j]}{P[H_i]}\right) \text{ for any } j = 0, \ldots, M-1.$$

Example: M-ary MAP decision rule for time-limited discrete-time signals

$$\text{Select } H_i \text{ if}$$
$$\boldsymbol{y}^T\boldsymbol{s}_i + \sigma^2\ln(P[H_i]) - \frac{1}{2}E_{\boldsymbol{s}_i} \geq \boldsymbol{y}^T\boldsymbol{s}_j + \sigma^2\ln(P[H_j]) - \frac{1}{2}E_{\boldsymbol{s}_j}$$
$$\text{for any } j = 0, \ldots, M-1.$$

Compact formulation of the MAP decision rule:

$$\hat{H} = H_{\hat{i}} \text{ where}$$
$$\hat{i} = \operatorname{argmax}_i\left\{\boldsymbol{y}^T\boldsymbol{s}_i + \sigma^2\ln(P[H_i]) - \frac{1}{2}E_{\boldsymbol{s}_i}\right\}$$

Special case: ML decision rule:

Selecting the uniform "a priori" pdf

$$P[H_0] = P[H_1] = \ldots = P[H_{M-1}] = \frac{1}{M}$$

yields the ML decision rule:

$$\hat{H} = H_{\hat{i}} \text{ where}$$
$$\hat{i} = \operatorname{argmax}_i\left\{\boldsymbol{y}^T\boldsymbol{s}_i - \frac{1}{2}E_{\boldsymbol{s}_i}\right\}$$

Comment: The M-ary MAP decision rules for all other situations previously considered are obtained by appropriately replacing the scalar product in the above decision rules.

- ***Block diagram of a M-ary MAP detector:***