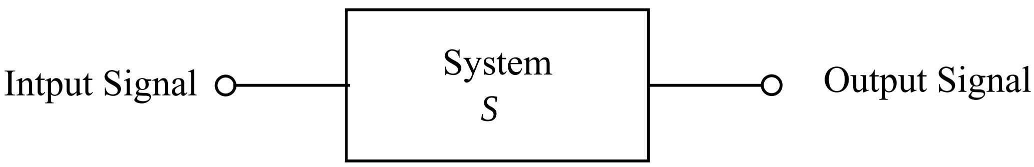
5. Response of Linear Time-Invariant Systems to Random Inputs

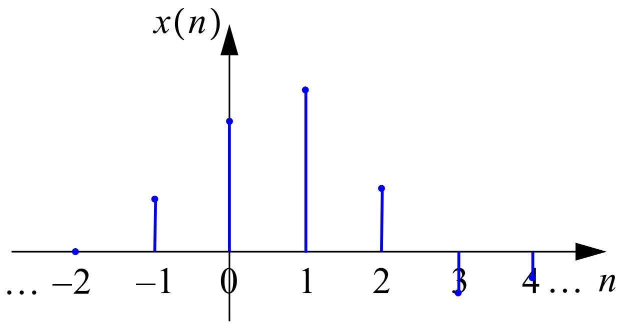
System:



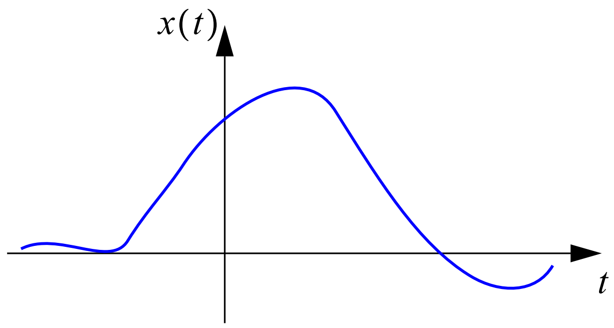
We look at a system as a black box which generates an output signal depending on the input signal and possibly some initial conditions.

We consider two types of signals:

- ***Discrete-time signals or sequences***



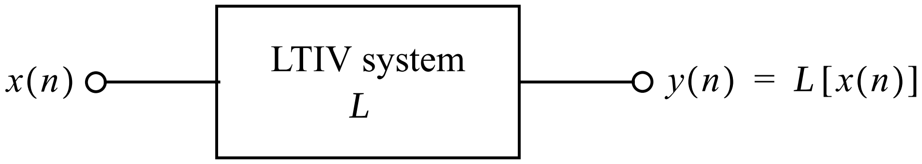
- ***Continuous-time signals***



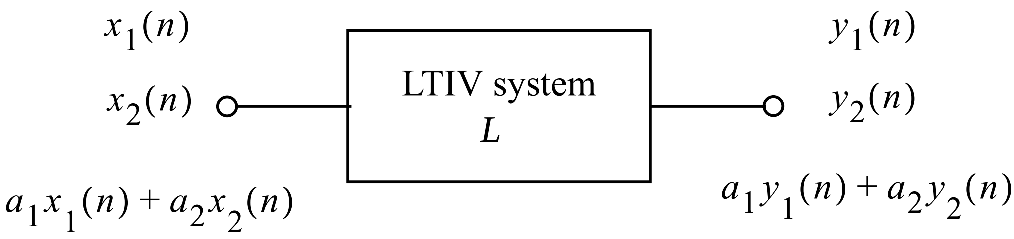
Discrete-time signals are obtained by sampling continuous-time signals.

5.1. Discrete-time linear time-invariant (LTIV) systems

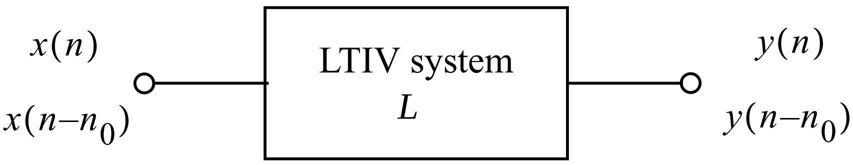
5.1.1. Discrete-time LTIV system



Linear:



Time-invariant:



5.1.2. Steady-state description of a LTIV system

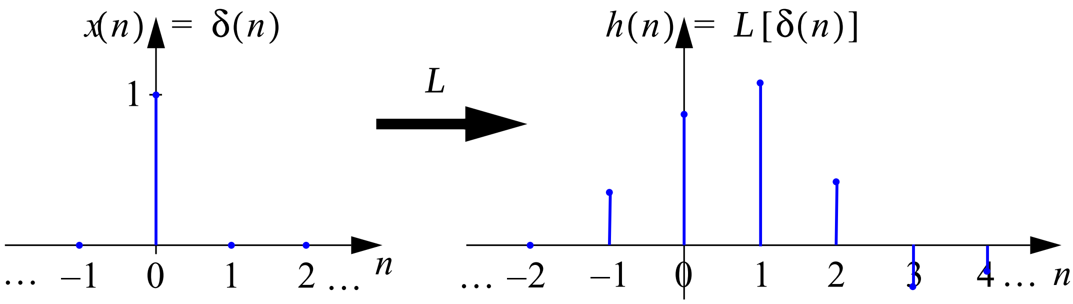
- ***Impulse response:***

  The impulse response (IR) $h(n)$ of L is the response of L to the unit pulse

$$\delta(n) \equiv \begin{cases} 1 \ ; & n = 0 \\ 0 \ ; & n \neq 0 \end{cases} ,$$

namely

$$h(n) = L[\delta(n)]$$



- ***Stable LTIV system:***
  A LTIV system is stable if its response to a bounded signal is bounded.

  It can be shown [to this end we need (5.1)] that a LTIV system is stable if, and if,

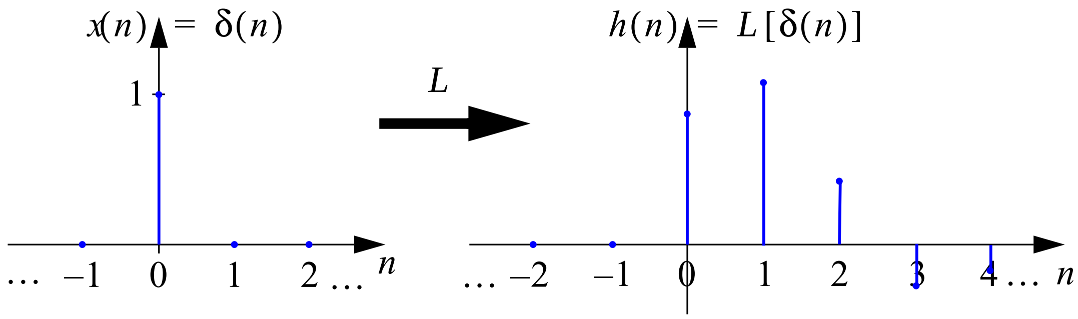
$$\sum_{m=-\infty}^{\infty} |h(m)| < \infty$$

Proof:

□

- ***Causal LTIV system:***

$$h(n) = 0 \qquad \text{for} \qquad n < 0$$



- ***Input-output (I-O) relationship of a LTIV system (time domain):***

$$y(n) = \sum_{m=-\infty}^{\infty} h(m)x(n-m) = h(n) * x(n) \qquad (5.1)$$

The symbol * denotes the discrete convolution operation.
Proof:

□

- ***(Discrete) Fourier transform:***
  Here, $z(n)$ denotes an arbitrary sequence.

$$Z(f) = F\{z(n)\} \equiv \sum_{n=-\infty}^{\infty} z(n)\exp(-j2\pi nf) \qquad (|f| < 1/2)$$

$$z(n) = F^{-1}\{Z(f)\} = \int_{-1/2}^{1/2} Z(f)\exp(j2\pi nf)df$$

Useful property extensively used in the sequel:

$$z(n-n_0) \quad \circ\!\!-\!\!\!-\!\!\bullet \quad \exp(-j2\pi n_0 f)Z(f)$$

Proof:

$\square$

- ***(Frequency) transfer function of a LTIV system:***

$$H(f) \equiv F\{h(n)\} \equiv \sum_{n=-\infty}^{\infty} h(n)\exp(-j2\pi nf)$$

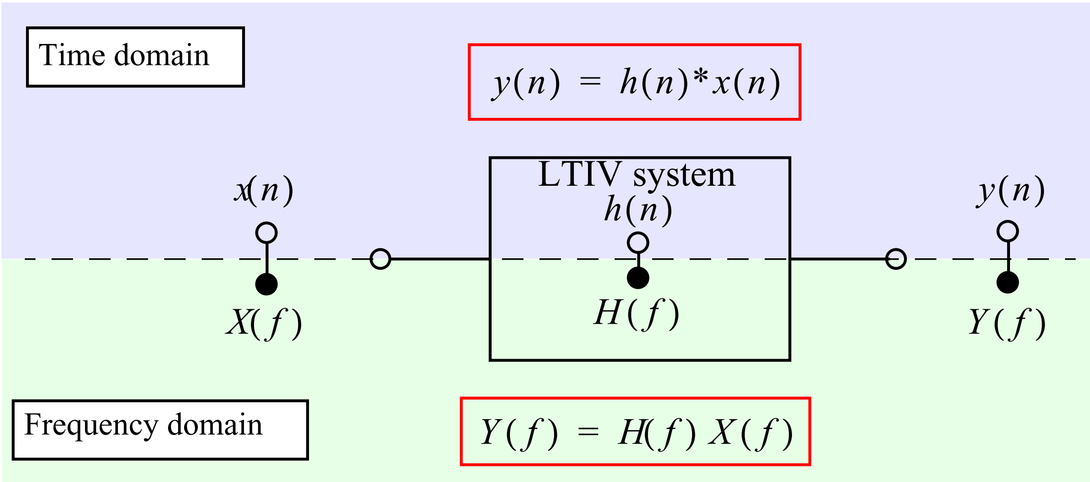
- ***I-O relationship of a LTIV system (frequency domain):***

$$Y(f) = H(f)X(f)$$

Proof:

$\square$

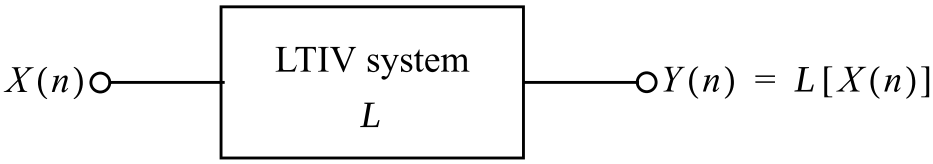
- ***Summary: I-O relationship of a LTIV system:***



5.1.3. First- and second-order characterization of a LTVI system

- ***Random input and output sequences:***

  If $X(n)$ is a random sequence (or process), so is $Y(n)$.



- ***Second-order characterization of random sequences:***

  Here, $Z(n)$ denotes an arbitrary random sequence.

  - Expectation:

$$\mu_Z(n) \equiv \mathbf{E}[Z(n)]$$

  - Autocorrelation function:

$$R_{ZZ}(n_1, n_2) \equiv \mathbf{E}[Z(n_1)Z(n_2)]$$

- ***Second-order properties of the output sequence $Y(n)$:***

  - Expectation:

$$\mu_Y(n) = h(n)*\mu_X(n)$$

- Autocorrelation function:

$$R_{YY}(n_1, n_2) = \sum_{m_1 = -\infty}^{\infty} \sum_{m_2 = -\infty}^{\infty} h(m_1)h(m_2)R_{XX}(n_1 - m_1, n_2 - m_2)$$

Proof:

□

5.1.4. Wide-sense-stationary (WSS) processes

- ***Definition:***

  A random sequence $Z(n)$ is WSS if the following conditions are satisfied:

  - Expectation:

  $$\mu_Z(n) = \mathbf{E}[Z(n)] = \mu_Z$$

  - Autocorrelation function:

  $$R_{ZZ}(n_1, n_1 + k) = \mathbf{E}[Z(n_1)Z(n_1 + k)] = R_{ZZ}(k)$$

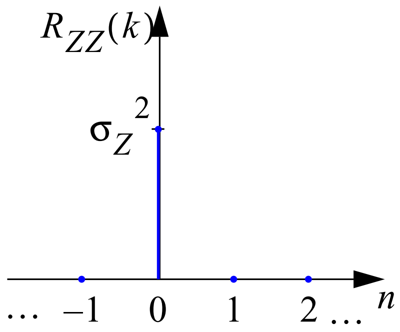
- ***White process:***

  $Z(n)$ is a white process if it satisfies the following conditions:

  - $Z(n)$ is a random process

  - $\mu_Z(n) = \mathbf{E}[Z(n)] = 0$

- $R_{ZZ}(n, n+k) = \mathbf{E}[Z(n)Z(n+k)] = R_{ZZ}(k) = \sigma_Z^2\delta(k)$



- ***Autocorrelation function of the impulse response:***

$$R_{hh}(k) = \sum_{m=-\infty}^{\infty} h(m)h(m+k)$$
$$= h(k)*h(-k)$$

Proof:

□

- ***Second-order I-O relationship of a LTVI system (time domain):***

  - Expectation:

$$\mu_Y = \left[\sum_{m=-\infty}^{\infty} h(m)\right]\mu_X = H(0)\mu_X$$

  - Autocorrelation function:

$$R_{YY}(k) = R_{hh}(k)*R_{XX}(k)$$

Proof:

□

- ***Power spectrum of a WSS process:***
  The power spectrum of the WSS process $Z(n)$ with the autocorrelation function $R_{ZZ}(k)$ is defined to be

$$S_{ZZ}(f) = F\{R_{ZZ}(k)\} = \sum_{n=-\infty}^{\infty} R_{ZZ}(k)\exp(-j2\pi kf)$$

  Notice that from the inverse Fourier transformation

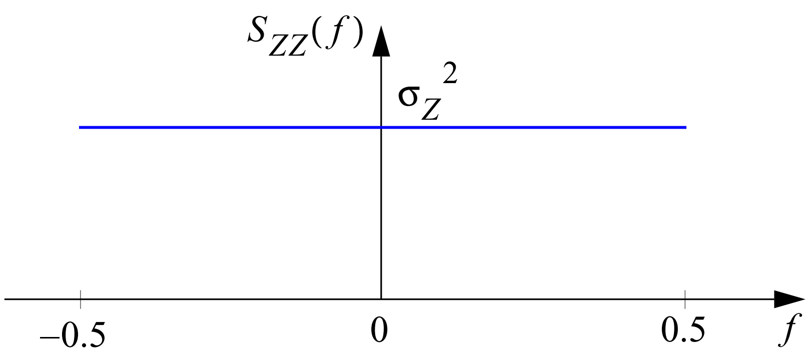
$$R_{ZZ}(k) = \int_{-\frac{1}{2}}^{\frac{1}{2}} S_{ZZ}(f)\exp(j2\pi kf)df$$

  we conclude that

$$\mathbf{E}[Z(n)^2] = R_{ZZ}(0) = \int_{-\frac{1}{2}}^{\frac{1}{2}} S_{ZZ}(f)df$$

- ***Spectrum of a white process:***
  If $Z(n)$ is a white process:

$$S_{ZZ}(f) = \sigma_Z^2$$



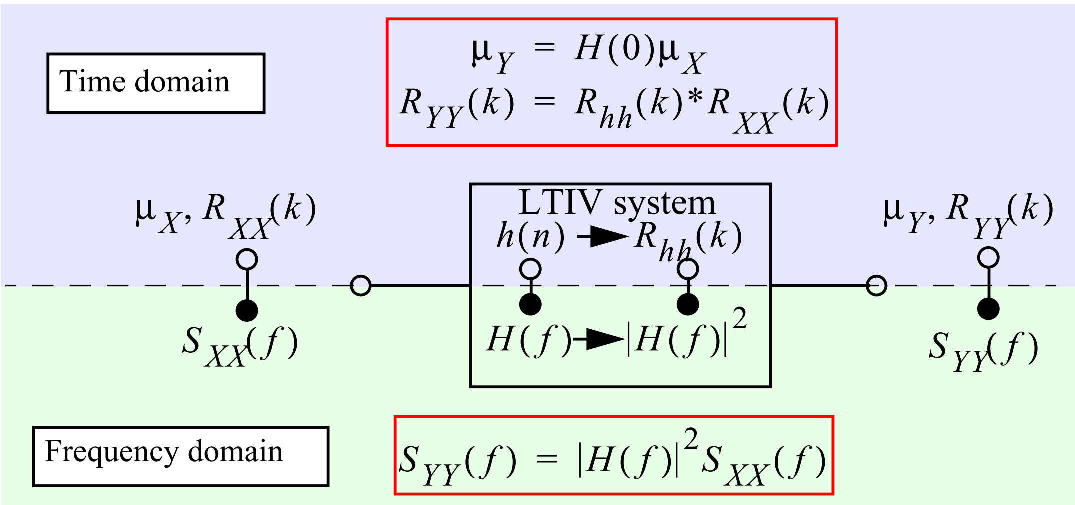
- ***Second-order I-O relationship of a LTVI system (frequency domain):***

$$\boxed{S_{YY}(f) = |H(f)|^2 S_{XX}(f)}$$

Proof:

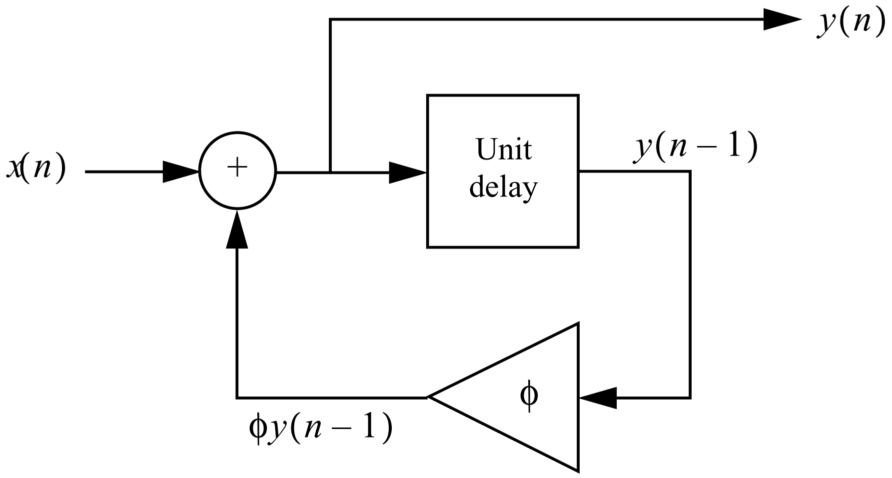
□

- ***Summary: Second-order I-O relationship of a LTIV system:***



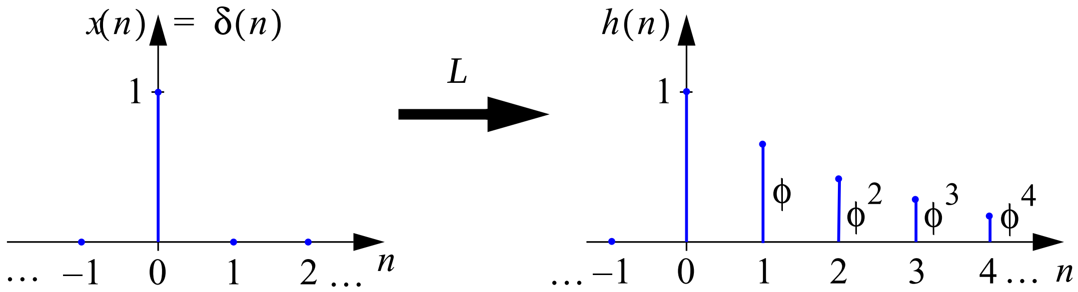
5.1.5. Example: First order recursive filter

- ***Block diagram and recursive equation:***



$$y(n) = x(n) + \phi y(n-1) \qquad (y(n) = 0 \qquad n < 0)$$

- ***Impulse response:***



$$h(n) = \begin{cases} 0 \; ; & n < 0 \\ \phi^n \; ; & n \geq 0 \end{cases}$$

- ***Stability condition:***

$$\sum_{n=-\infty}^{\infty} |h(n)| = \sum_{n=0}^{\infty} |\phi|^n = \lim_{N \to \infty} \frac{1-|\phi|^N}{1-|\phi|} < \infty \qquad \Leftrightarrow \qquad |\phi| < 1$$

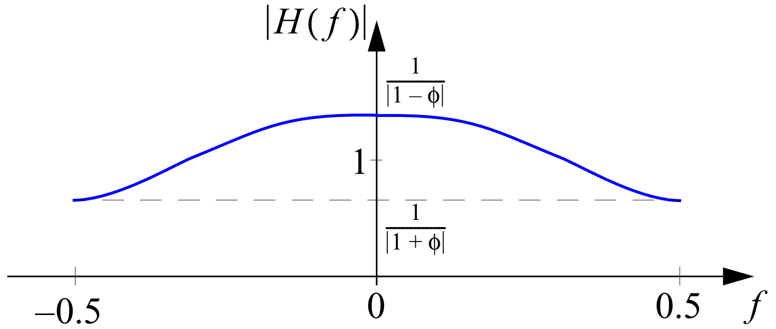
$$\left[\sum_{n=0}^{N} a^n = \frac{1-a^{N+1}}{1-a} \right]$$

- ***Transfer function:***

$$H(f) = F\{h(n)\} = \sum_{n=0}^{\infty} \phi^n \exp(-j2\pi nf)$$

$$= \sum_{n=0}^{\infty} [\phi \exp(-j2\pi f)]^n$$

$$= \frac{1}{1 - \phi \exp(-j2\pi f)}$$

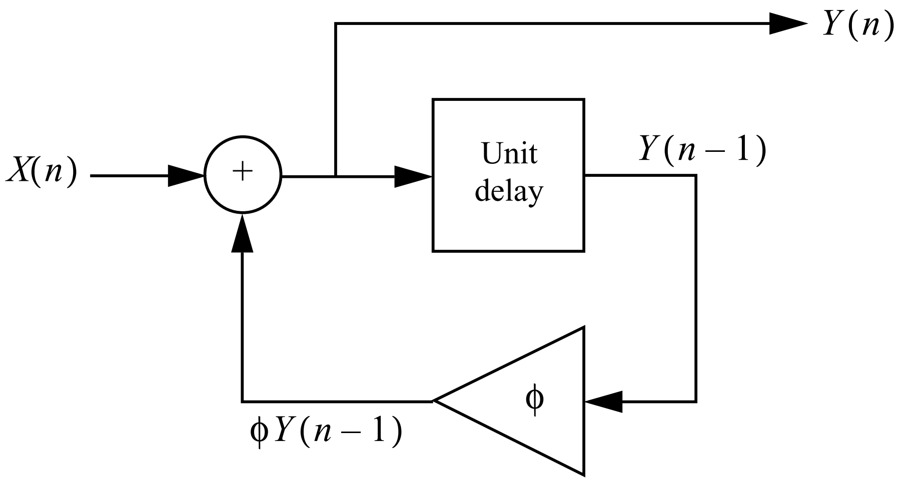


Another more direct way to compute the transfer function:

$$y(n) = x(n) + \phi y(n-1)$$

$$Y(f) = X(f) + \phi \exp(-j2\pi f) Y(f)$$

- ***Random input and output:***



$$Y(n) = X(n) + \phi Y(n-1)$$

- ***Second-order I-O relationship:***
  - Time domain:

$$\mu_Y = H(0)\mu_X = \frac{1}{1-\phi}\mu_X$$

$$R_{YY}(k) = R_{hh}(k) * R_{XX}(k) = \frac{\phi^{|k|}}{1-\phi^2} * R_{XX}(k)$$

$$\left[R_{hh}(k) = \sum_{m=0}^{\infty} \phi^m \phi^{m+|k|} = \phi^{|k|} \sum_{m=0}^{\infty} \phi^{2m} = \phi^{|k|} \frac{1}{1-\phi^2} \right]$$

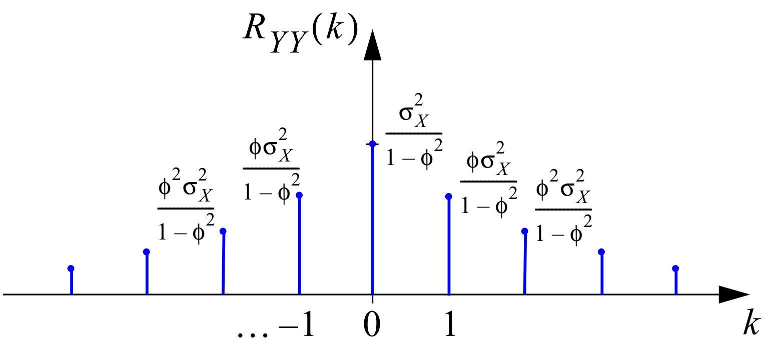
  - Frequency domain:

$$S_{YY}(f) = |H(f)|^2 S_{XX}(f) = \frac{1}{|1-\phi\exp(-j2\pi f)|^2} S_{XX}(f)$$

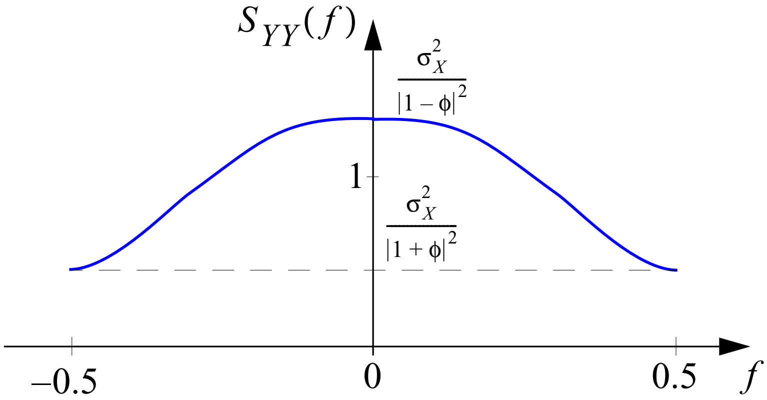
- ***Special case: AR(1) process (see Section 6.2):***
  If $X(n)$ is a white Gaussian process,

$$\mu_Y = 0$$

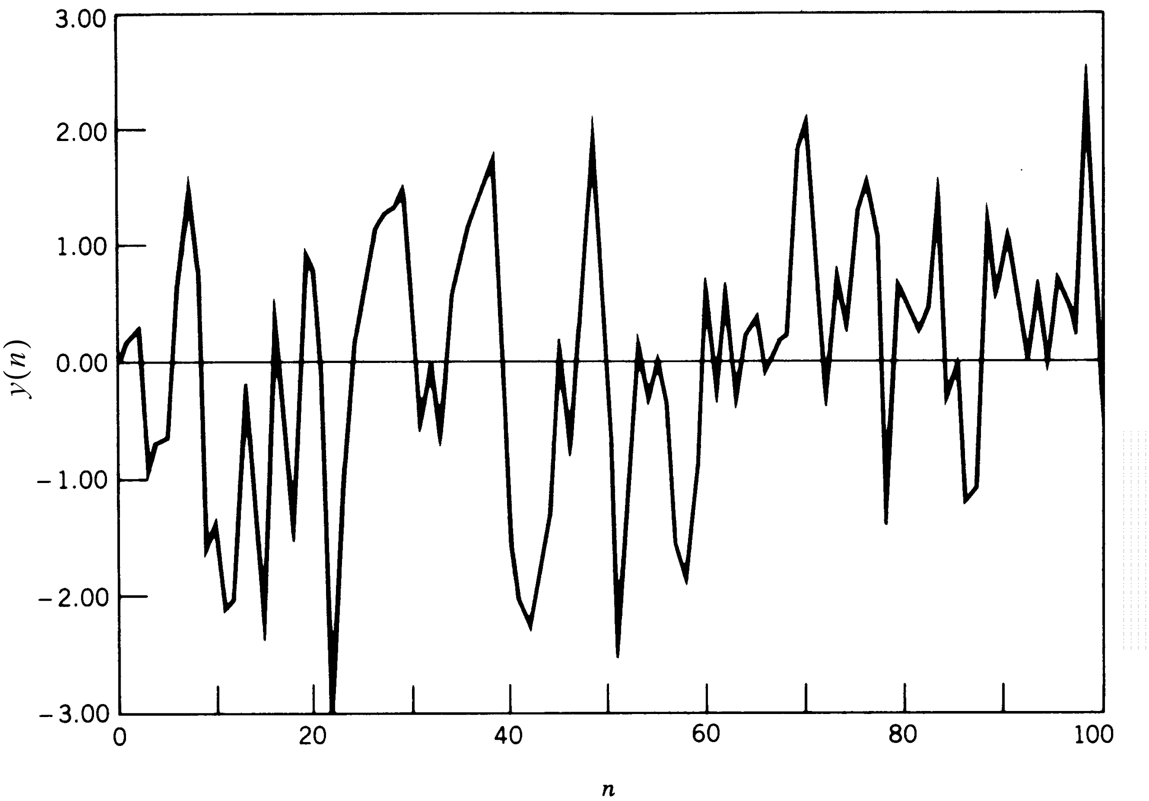
$$R_{YY}(k) = \frac{\phi^{|k|}}{1-\phi^2}\sigma_X^2$$



$$S_{YY}(f) = \frac{\sigma_X^2}{|1 - \phi \exp(-j2\pi f)|^2}$$

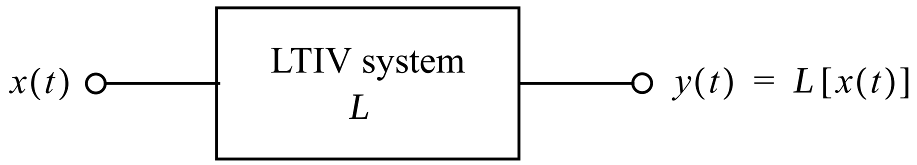


One realization of $Y(n)$:

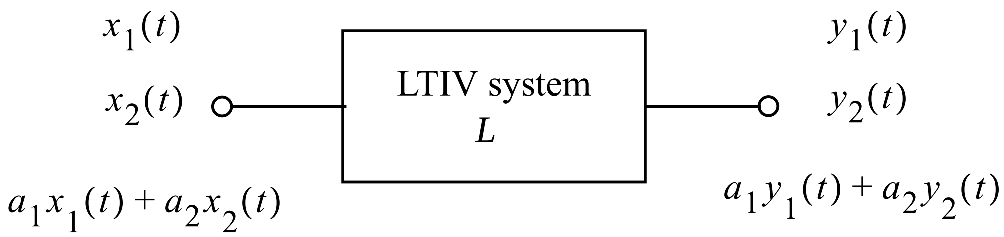


5.2. Continuous-time LTIV systems

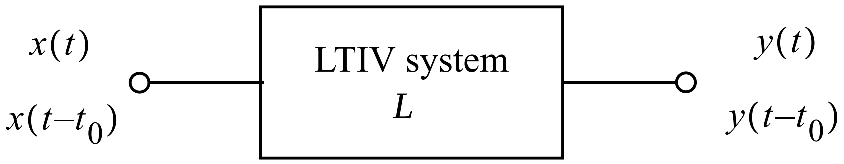
5.2.1. Continuous-time LTIV system



Linear:



Time-invariant:



5.2.2. Steady-state description of a LTIV system

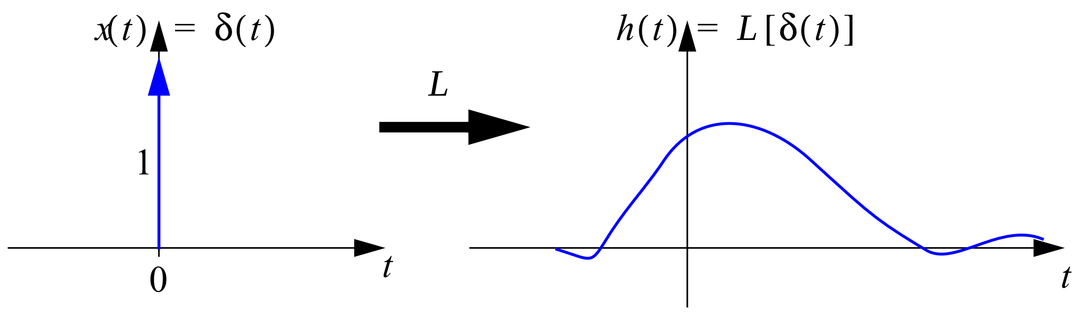
- ***Impulse response:***

  The impulse response $h(t)$ of L is the response of L to the Dirac impulse

$$\delta(t) \equiv \begin{cases} \infty \; ; & t = 0 \\ 0 \; ; & t \neq 0 \end{cases} \qquad \int_{-\infty}^{\infty} \delta(t)dt = 1,$$

namely

$$h(t) = L[\delta(t)]$$



- ***Stable LTIV system:***
  A LTIV system is stable if its response to a bounded signal is bounded.

  It can be shown [to this end we need (5.2)] that a LTIV system is stable if, and if,

$$\int_{-\infty}^{\infty} |h(t)| dt < \infty$$

- ***Causal LTIV system:***

$$h(t) = 0 \qquad t < 0$$

- ***Input-output relationship of a LTIV system (time domain):***

$$\boxed{\begin{aligned} y(t) &= \int_{-\infty}^{\infty} h(\tau)x(t-\tau)d\tau \\ &= h(t)*x(t) \end{aligned}} \qquad (5.2)$$

  Here, the symbol * denotes the continuous convolution operation.

- ***(Continuous) Fourier transform:***

$$Z(f) = F\{z(t)\} \equiv \int_{-\infty}^{\infty} z(t)\exp(-j2\pi ft)d\tau$$

$$\bullet\!\!-\!\!\circ$$

$$z(t) = F^{-1}\{Z(f)\} = \int_{-\infty}^{\infty} Z(f)\exp(j2\pi ft)df$$

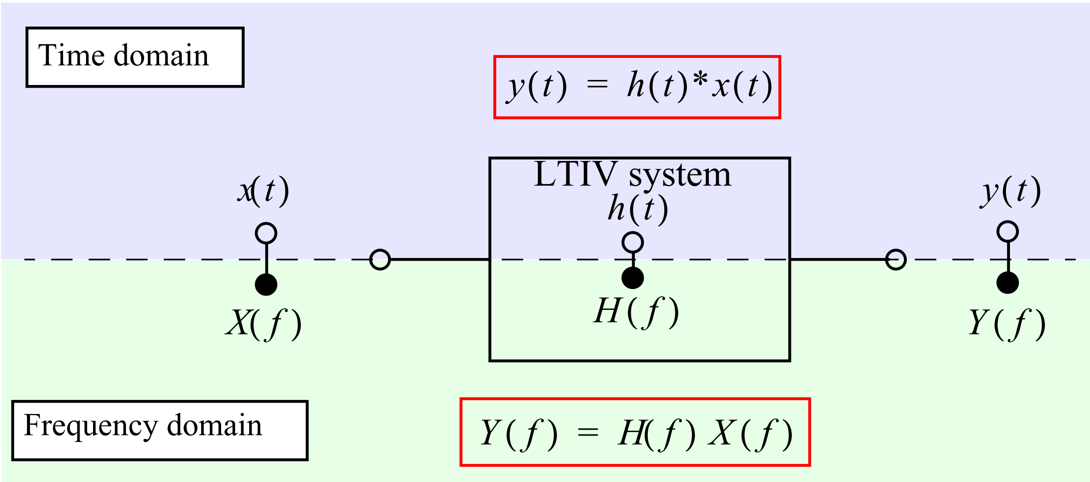
- ***(Frequency) transfer function of a LTIV system***

$$\boxed{H(f) = F\{h(t)\} = \int_{-\infty}^{\infty} h(t)\exp(-j2\pi ft)dt}$$

- ***I-O relation ship of a LTIV system (frequency domain):***

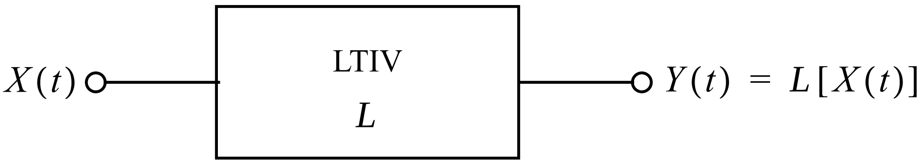
$$\boxed{Y(f) = H(f)X(f)}$$

- ***Summary: I-O relation ship of a LTIV system:***



5.2.3. Second-order characterization of LTIV

- ***Random input and output processes:***



- ***Second-order characterization of random processes:***

  Here, $Z(t)$ denotes an arbitrary random process.

  - Expectation:

$$\mu_Z(t) \equiv \mathbf{E}[Z(t)]$$

  - Autocorrelation function:

$$R_{ZZ}(t_1, t_2) \equiv \mathbf{E}[Z(t_1)Z(t_2)]$$

- ***Second-order properties of the output process*** $Y(t)$***:***

  - Expectation:

$$\mu_Y(t) = h(t)*\mu_X(t)$$

  - Autocorrelation function:

$$R_{YY}(t_1, t_2) \equiv \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} h(u_1)h(u_2)R_{XX}(t_1 - u_1, t_2 - u_2)du_1 du_2$$

5.2.4. Wide-sense-stationary (WSS) processes

- ***Wide-sense stationarity:***

  A random process $Z(t)$ is WSS if the following conditions are satisfied:

  - Expectation:

$$\mu_Z(t) \equiv \mu_Z$$

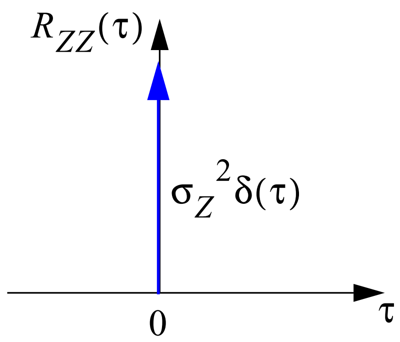
  - Autocorrelation function:

$$R_{ZZ}(t_1, t_1 + \tau) \equiv R_{ZZ}(\tau)$$

- ***White process:***

  $Z(t)$ is a white process if it satisfies the following conditions:

  - $Z(t)$ is a random process
  - $\mu_Z(t) = \mathbf{E}[Z(t)] = 0$
  - $R_{ZZ}(t, t+\tau) = \mathbf{E}[Z(t)Z(t+\tau)] = R_{ZZ}(\tau) = \sigma_Z^2\delta(\tau)$



- ***Autocorrelation function of the impulse response:***

$$R_{hh}(\tau) = \int_{-\infty}^{\infty} h(u)h(u+\tau)du$$
$$= h(\tau)*h(-\tau)$$

- ***Second-moment I-O relationship of a LTVI system (time domain):***

  - Expectation:

$$\mu_Y = \left[\int_{-\infty}^{\infty} h(u)du\right]\mu_X = H(0)\mu_X$$

- Autocorrelation function:

$$R_{YY}(\tau) = R_{hh}(\tau) * R_{XX}(\tau)$$

- ***Power spectrum of a WSS process:***
  $Z(t)$ is WSS with the autocorrelation function $R_{ZZ}(\tau)$.

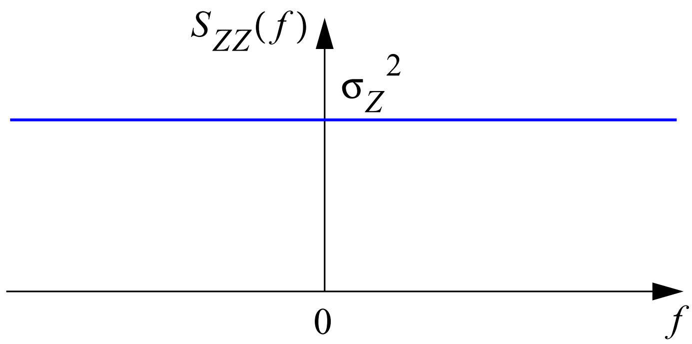
$$S_{ZZ}(f) = F\{R_{ZZ}(\tau)\}$$

  Notice the identities

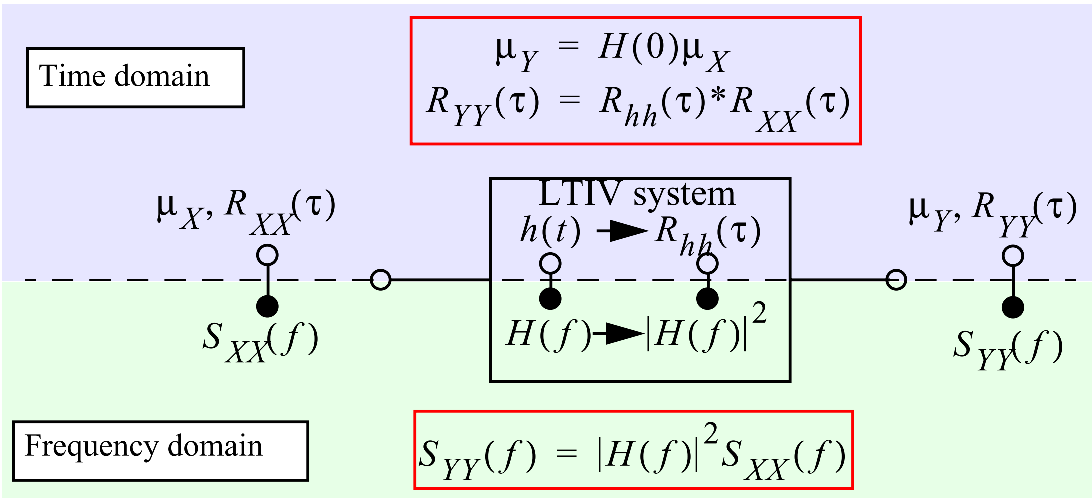
$$\mathbf{E}[Z(t)^2] = R_{ZZ}(0) = \int_{-\infty}^{\infty} S_{ZZ}(f)df$$

- ***Spectrum of a white process:***
  If $Z(t)$ is a white process:

$$S_{ZZ}(f) = \sigma_Z^2$$

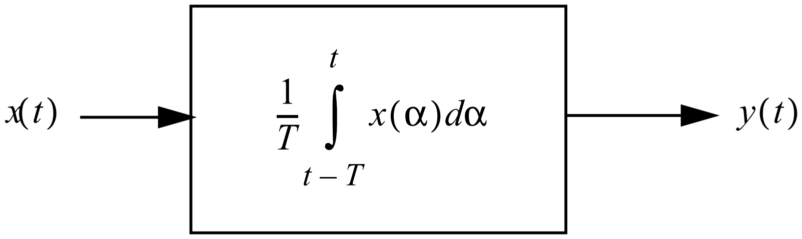


- ***Summary: Second order I-O relationship of a LTIV system:***



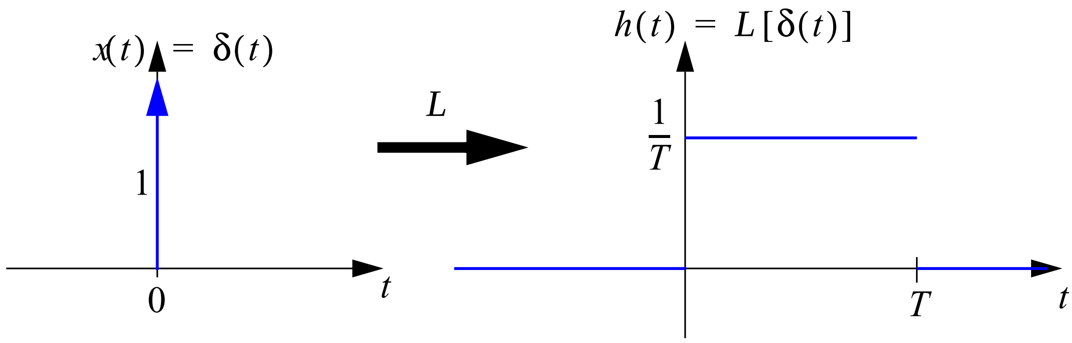
5.2.5. Example: Ideal integrator

- ***Block diagram and input-output relationship:***



$$y(t) = \frac{1}{T}\int_{t-T}^{t} x(\alpha)\,d\alpha$$

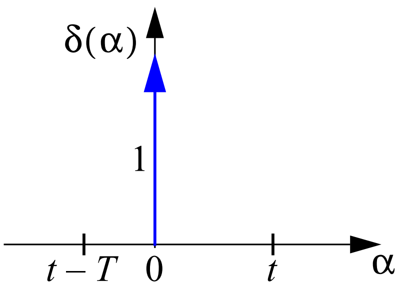
- ***Impulse response:***



$$h(t) = \begin{cases} \frac{1}{T} \; ; & 0 < t < T \\ 0 \; ; & \text{elsewhere} \end{cases}$$

Proof:

$$\frac{1}{T}\int_{t-T}^{t} \delta(\alpha)\,d\alpha = \begin{cases} 1 \; ; & 0 \in (t, t-T) \\ 0 \; ; & \text{elsewhere} \end{cases}$$



□

- ***Stability condition:***

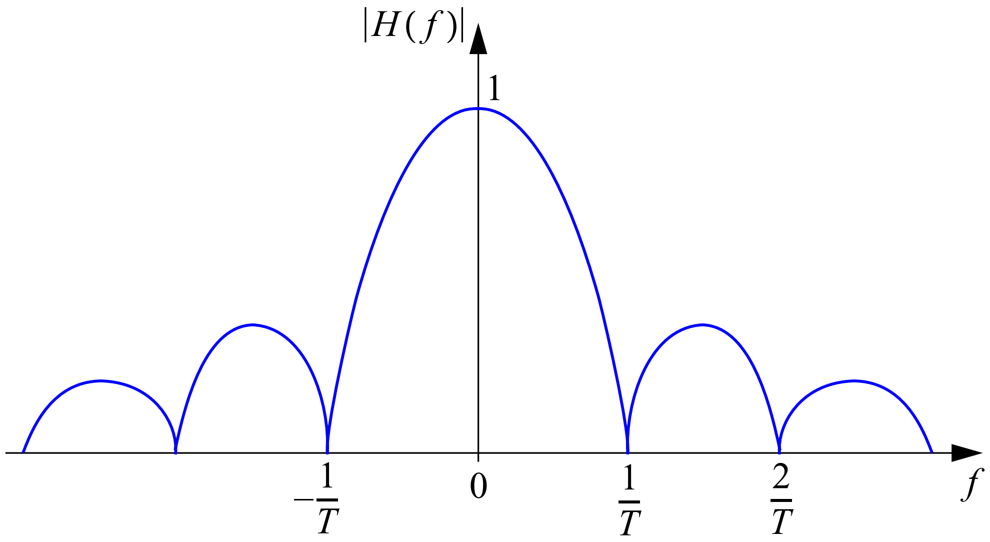
$$\int_{-\infty}^{\infty} |h(t)|\, dt = 1$$

- ***Transfer function:***

$$H(f) = F\{h(t)\} = \int_{-\infty}^{\infty} h(t)\exp(-j2\pi ft)\,dt$$

$$= \frac{1}{T}\int_{0}^{T} \exp(-j2\pi ft)\,dt = \exp(-j\pi fT)\frac{\sin(\pi fT)}{\pi fT}$$

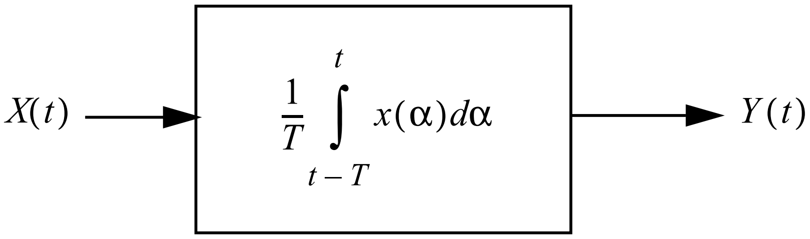
$$= \exp(-j\pi fT)\,\mathrm{sinc}(fT)$$



Proof:

□

- ***Random input and output:***



$$Y(t) = \frac{1}{T}\int_{t-T}^{t} X(\alpha)\,d\alpha$$

- ***Second-order I-O relationship:***

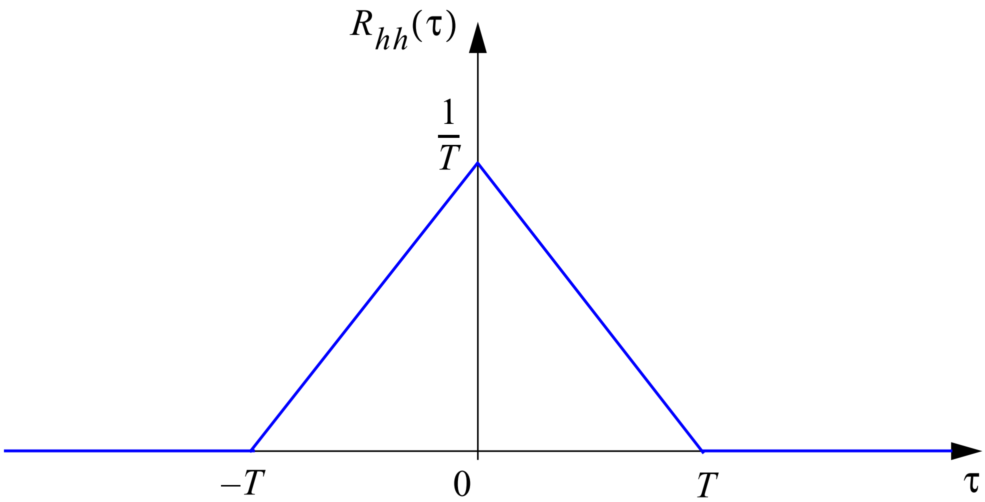
  - Time domain:

$$\mu_Y = H(0)\mu_X$$
$$= \mu_X$$

$$R_{YY}(\tau) = R_{hh}(\tau) * R_{XX}(\tau)$$

with

$$R_{hh}(\tau) = \begin{cases} \frac{1}{T}\left(1 - \frac{|\tau|}{T}\right) & ; \quad -T < \tau < T \\ 0 & ; \quad \text{elsewhere} \end{cases}$$



Proof:

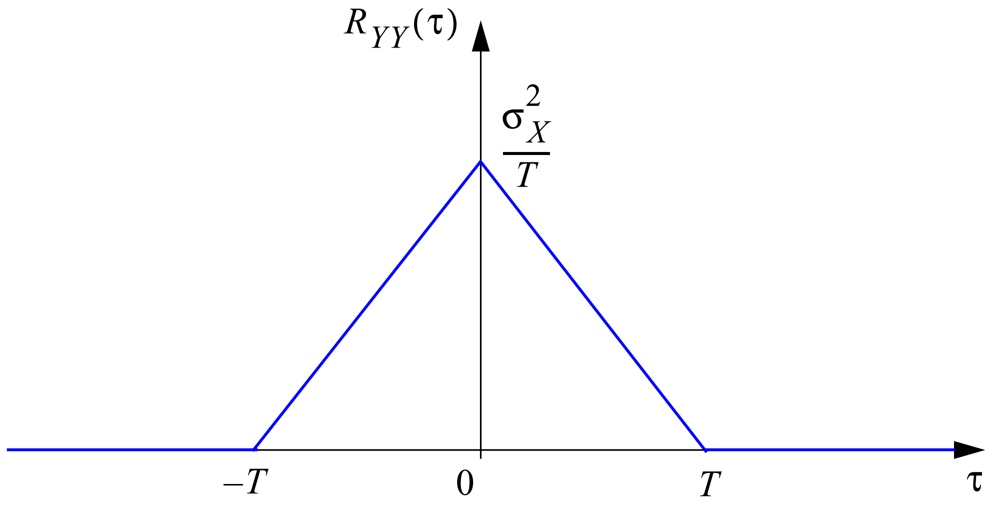
□

- Frequency domain:

$$S_{YY}(f) = |H(f)|^2 S_{XX}(f)$$
$$= \mathrm{sinc}(fT)^2 S_{XX}(f)$$

- ***Special case:*** $X(t)$ ***is a white process,***

$$\mu_Y = 0 \qquad\qquad R_{YY}(\tau) = \begin{cases} \dfrac{\sigma_X^2}{T}\left(1 - \dfrac{|\tau|}{T}\right) & ; \quad -T < \tau < T \\ 0 & ; \quad \text{elsewhere} \end{cases}$$



$$S_{YY}(f) = \sigma_X^2 \operatorname{sinc}(fT)^2$$