# 2. Linear Minimum Mean Squared Error Estimation

## 2.1. Linear minimum mean squared error estimators

- ***Situation considered:***

  - A random sequence $X(1), \ldots, X(M)$ whose realizations can be observed.

  - A random variable $Y$ which has to be estimated.

  - We seek an estimate of $Y$ with a **linear estimator** of the form:

$$\boxed{\hat{Y} = h_0 + \sum_{m=1}^{M} h_m X(m)}\ .$$

  - A measure of the goodness of $\hat{Y}$ is the mean squared error (MSE):

$$\boxed{\mathbf{E}[(\hat{Y} - Y)^2]}\ .$$

- ***Covariance and variance of random variables:***
  Let $U$ and $V$ denote two random variables with expectation $\mu_U \equiv \mathbf{E}[U]$ and $\mu_V \equiv \mathbf{E}[V]$.

  - The **covariance** of $U$ and $V$ is defined to be:

$$\Sigma_{UV} \equiv \mathbf{E}[(U - \mu_U)(V - \mu_V)]$$
$$= \mathbf{E}[UV] - \mu_U \mu_V$$

  - The **variance** of $U$ is defined to be:

$$\sigma_U^2 \equiv \mathbf{E}[(U - \mu_U)^2] = \Sigma_{UU}$$
$$= \mathbf{E}[U^2] - (\mu_U)^2$$

Let $\boldsymbol{U} \equiv [U(1), \ldots, U(M)]^T$ and $\boldsymbol{V} \equiv [V(1), \ldots, V(M')]^T$ denote two random vectors.
The covariance matrix of $\boldsymbol{U}$ and $\boldsymbol{V}$ is defined as

$$\Sigma_{\boldsymbol{UV}} \equiv \begin{bmatrix} \Sigma_{U(1)V(1)} & \cdots & \Sigma_{U(1)V(M')} \\ \cdots & \cdots & \cdots \\ \Sigma_{U(M)V(1)} & \cdots & \Sigma_{U(M)V(M')} \end{bmatrix}$$

A direct way to obtain $\Sigma_{\boldsymbol{UV}}$:

$$\begin{aligned} \Sigma_{\boldsymbol{UV}} &= \mathbf{E}[(\boldsymbol{U} - \mu_{\boldsymbol{U}})(\boldsymbol{V} - \mu_{\boldsymbol{V}})^T] \\ &= \mathbf{E}[\boldsymbol{U}\boldsymbol{V}^T] - \mu_{\boldsymbol{U}}(\mu_{\boldsymbol{V}})^T \end{aligned}$$

where

$$\begin{aligned} \mu_{\boldsymbol{U}} &\equiv \mathbf{E}[\boldsymbol{U}] = [\mathbf{E}[U(1)], \ldots, \mathbf{E}[U(M)]]^T \\ \mu_{\boldsymbol{V}} &\equiv \mathbf{E}[\boldsymbol{V}] \end{aligned}$$

**Examples:** $\boldsymbol{U} = \boldsymbol{X} \equiv [X(1), \ldots, X(M)]^T$ and $\boldsymbol{V} = Y$.
In the sequel we shall frequently make use of the following covariance matrix and vector:

(i) $$\Sigma_{\boldsymbol{XX}} = \mathbf{E}[(\boldsymbol{X} - \mu_{\boldsymbol{X}})(\boldsymbol{X} - \mu_{\boldsymbol{X}})^T] = \begin{bmatrix} \sigma^2_{X(1)} & \cdots & \Sigma_{X(1)X(M)} \\ \cdots & \cdots & \cdots \\ \Sigma_{X(M)X(1)} & \cdots & \sigma^2_{X(M)} \end{bmatrix}$$

(ii) $$\begin{aligned} \Sigma_{\boldsymbol{X}Y} &= \mathbf{E}[(\boldsymbol{X} - \mu_{\boldsymbol{X}})(Y - \mu_Y)] \\ &= \begin{bmatrix} \Sigma_{X(1)Y} & \cdots & \Sigma_{X(M)Y} \end{bmatrix}^T \end{aligned}$$

- ***Linear minimum mean squared error estimator (LMMSEE)***
  A LMMSEE of $Y$ is a linear estimator, i.e. an estimator of the form

$$\hat{Y} = h_0 + \sum_{m=1}^{M} h_m X(m) \quad ,$$

which minimizes the MSE $\mathbf{E}[(\hat{Y} - Y)^2]$.

A linear estimator is entirely determined by the $(M+1)$-dimensional vector $\boldsymbol{h} \equiv [h_0, \ldots, h_M]^T$.

- ***Orthogonality principle:***

**Orthogonality principle:**
A necessary condition for $\boldsymbol{h} \equiv [h_0, \ldots, h_M]^T$ to be the coefficient vector of the LMMSEE is that its entries fulfils the $(M+1)$ identities:

$$\mathbf{E}[Y - \hat{Y}] = \mathbf{E}\left[Y - \left(h_0 + \sum_{m=1}^{M} h_m X(m)\right)\right] = 0 \qquad (2.1a)$$

$$\mathbf{E}[(Y - \hat{Y})X(j)] = \mathbf{E}\left[\left\{Y - \left(h_0 + \sum_{m=1}^{M} h_m X(m)\right)\right\}X(j)\right] = 0, \qquad (2.1b)$$

$$j = 1, \ldots, M$$

***Proof:***

Because the coefficient vector of the LMMSEE minimizes $\mathbf{E}[(\hat{Y} - Y)^2]$, its components must satisfy the set of equations:

$$\frac{\partial}{\partial h_j}\mathbf{E}[(\hat{Y} - Y)^2] = 0 \qquad j = 0, \ldots, M.$$

□

Notice that the two expressions in (2.1) can be rewritten as:

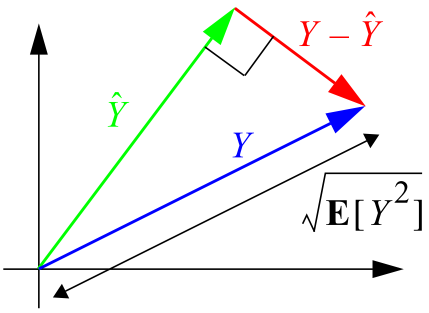
$$\mathbf{E}[Y - \hat{Y}] = 0 \qquad (2.2a)$$

$$\mathbf{E}[(Y - \hat{Y})X(j)] = 0 \qquad j = 1, \ldots, M \qquad (2.2b)$$

**Important consequences of the orthogonality principle:**

$$\mathbf{E}[(Y - \hat{Y})\hat{Y}] = 0 \qquad (2.3a)$$

$$\mathbf{E}[(Y - \hat{Y})^2] = \mathbf{E}[Y^2] - \mathbf{E}[\hat{Y}^2] = \mathbf{E}[(Y - \hat{Y})Y] \qquad (2.3b)$$



**Geometrical interpretation:**
Let $U$ and $V$ denote two random variables with finite second moment, i.e. $\mathbf{E}[U^2] < \infty$ and $\mathbf{E}[V^2] < \infty$.
Then, the expectation $\mathbf{E}[UV]$ can be viewed as the **scalar or inner product** of $U$ and $V$.

Within this interpretation:

- $U$ and $V$ are uncorrelated, i.e. $\mathbf{E}[UV] = 0$ if and only if, they are **orthogonal**,
- $\sqrt{\mathbf{E}[U^2]}$ is the norm (length) of $U$.

Interpretation of both equations in (2.3):

- (2.3a): the estimation error $Y - \hat{Y}$ and the estimate $\hat{Y}$ are orthogonal.
- (2.3b): results from Pythagoras' Theorem.

• ***Computation of the coefficient vector of the LMMSEE:***
The coefficients of the LMMSEE satisfy the relationships:

$$\mu_Y = h_0 + \sum_{m=1}^{M} h_m \mu_{X(m)} = h_0 + (\boldsymbol{h}^-)^T \mu_{\boldsymbol{X}}$$

$$\Sigma_{\boldsymbol{X}Y} = \Sigma_{\boldsymbol{XX}} \boldsymbol{h}^-$$

where $\boldsymbol{h}^- \equiv [h_1, \ldots, h_M]^T$ and $\boldsymbol{X} \equiv [X_1, \ldots, X_M]^T$.

***Proof:***
Both identities follow by appropriately reformulating relations (2.1a) and (2.1b) and using a matrix notation for the latter one.

□

Thus, provided $(\Sigma_{\boldsymbol{XX}})^{-1}$ exists the coefficients of the LMMSEE are given by:

$$\bar{\boldsymbol{h}} = (\Sigma_{\boldsymbol{XX}})^{-1}\Sigma_{\boldsymbol{X}Y} \qquad (2.4a)$$

$$h_0 = \mu_Y - (\bar{\boldsymbol{h}})^T \mu_{\boldsymbol{X}} = \mu_Y - \Sigma_{\boldsymbol{X}Y}{}^T (\Sigma_{\boldsymbol{XX}})^{-1} \mu_{\boldsymbol{X}} \qquad (2.4b)$$

- ***Example: Linear prediction of a WSS process***

  Let $Y(n)$ denote a WSS process with

  - zero mean, i.e $\mathbf{E}[Y(n)] = 0$,
  - autocorrelation function $\mathbf{E}[Y(n)Y(n+k)] = R_{YY}(k)$

  We seek the LMMSEE for the present value of $Y(n)$ based on the $M$ past observations $Y(n-1), \ldots, Y(n-M)$ of the process. Hence,

  - $Y = Y(n)$
  - $X(m) = Y(n-m)$, $m = 1, \ldots, M$, i.e.

    $\boldsymbol{X} = [Y(n-1), \ldots, Y(n-M)]^T$

  Because $\mu_Y = 0$ and $\mu_{\boldsymbol{X}} = 0$, it follows from $(2.4b)$ that

$$h_0 = 0$$

  Computation of $\Sigma_{\boldsymbol{X}Y}$ and $\Sigma_{\boldsymbol{XX}}$:

$$\text{-}\ \Sigma_{\boldsymbol{X}Y} = [\mathbf{E}[Y(n-1)Y(n)], \ldots, \mathbf{E}[Y(n-M)Y(n)]]^T$$

$$= [R_{YY}(1), \ldots, R_{YY}(M)]^T$$

$$\text{-}\ \Sigma_{\boldsymbol{XX}} =$$

$$= \begin{bmatrix} \mathbf{E}[Y(n-1)^2] & \mathbf{E}[Y(n-1)Y(n-2)] & \ldots & \mathbf{E}[Y(n-1)Y(n-M)] \\ \mathbf{E}[Y(n-2)Y(n-1)] & \mathbf{E}[Y(n-2)^2] & \ldots & \mathbf{E}[Y(n-2)Y(n-M)] \\ \ldots & \ldots & \ldots & \ldots \\ \mathbf{E}[Y(n-M)Y(n-1)] & \mathbf{E}[Y(n-M)Y(n-2)] & \ldots & \mathbf{E}[Y(n-M)^2] \end{bmatrix}$$

$$= \begin{bmatrix} R_{YY}(0) & R_{YY}(1) & R_{YY}(2) & \ldots & R_{YY}(M-1) \\ R_{YY}(1) & R_{YY}(0) & R_{YY}(1) & \ldots & R_{YY}(M-2) \\ R_{YY}(2) & R_{YY}(1) & R_{YY}(0) & \ldots & R_{YY}(M-3) \\ \ldots & \ldots & \ldots & \ldots & \ldots \\ R_{YY}(M-1) & R_{YY}(M-2) & R_{YY}(M-3) & \ldots & R_{YY}(0) \end{bmatrix}$$

- ***Residual error using a LMMSEE:***
  The MSE resulting when using a LMMSEE is

$$\mathbf{E}[(\hat{Y} - Y)^2] = \sigma_Y^{\ 2} - (\boldsymbol{h}^-)^T \Sigma_{\boldsymbol{X}Y} \qquad (2.5)$$

***Proof:***

$$\begin{aligned}\mathbf{E}[(Y - \hat{Y})^2] &= \mathbf{E}[(Y - \hat{Y})Y] \\ &= \mathbf{E}[Y^2] - \mathbf{E}[\hat{Y}Y]\end{aligned}$$

□

## 2.2. Minimum mean squared error estimators

- ***Conditional expectation:***
  Let $U$ and $V$ denote two random variables.
  The conditional expectation of $V$ given $U = u$ is observed is defined to be

$$\mathbf{E}[V|u] \equiv \int vp(v|u)dv .$$

Notice that $\mathbf{E}[V|U]$ is a random variable. In the sequel we shall make use of the following important property of conditional expectations:

$$\mathbf{E}[\mathbf{E}[V|U]] = \mathbf{E}[V]$$

***Proof:***

□

- ***Minimum mean squared error estimator (MMSEE):***

The MMSEE of $Y$ based on the observation of $X(1), \ldots, X(M)$ is of the form:

$$\boxed{\widehat{Y}(X(1), \ldots, X(M)) = \mathbf{E}[Y|X(1), \ldots, X(M)]}$$

Hence if $X(1) = x(1), \ldots, X(M) = x(M)$ is observed, then

$$\begin{aligned}\widehat{Y}(x(1), \ldots, x(M)) &= \mathbf{E}[Y|x(1), \ldots, x(M)] \\ &= \int y p(y|x(1), \ldots, x(M)) dy\end{aligned}$$

***Proof:***

Let $\hat{Y}$ denote an arbitrary estimator. Then,

$$\begin{aligned}\mathbf{E}[(\hat{Y} - Y)^2] &= \mathbf{E}[((\hat{Y} - \widehat{Y}) - (Y - \widehat{Y}))^2] \\ &= \mathbf{E}[(\hat{Y} - \widehat{Y})^2] - \underbrace{2\mathbf{E}[(\hat{Y} - \widehat{Y})(Y - \widehat{Y})]}_{= 0} + \mathbf{E}[(Y - \widehat{Y})^2] \\ &= \mathbf{E}[(\hat{Y} - \widehat{Y})^2] + \mathbf{E}[(Y - \widehat{Y})^2]\end{aligned}$$

Thus,

$$\mathbf{E}[(\hat{Y} - Y)^2] \geq \mathbf{E}[(Y - \widehat{Y})^2]$$

with equality if, and only if, $\hat{Y} = \widehat{Y}$. We still have to prove that $\mathbf{E}[(\hat{Y} - \widehat{Y})(Y - \widehat{Y})] = 0$.

□

**Example: Multivariate Gaussian variables:**

$[Y, X(1), \ldots, X(M)]^T \sim N(\mu, \Sigma)$ with

- $\mu \equiv [\mu_Y, \mu_{X(1)}, \ldots, \mu_{X(M)}]^T$

- $\Sigma \equiv \begin{bmatrix} \sigma_Y^2 & (\Sigma_{\boldsymbol{X}Y})^T \\ \Sigma_{\boldsymbol{X}Y} & \Sigma_{\boldsymbol{XX}} \end{bmatrix}$

From Equation (6.22) in [Shanmugan] it follows that

$$\widehat{Y} = \mathbf{E}[Y|\boldsymbol{X}] = \mu_Y + (\Sigma_{\boldsymbol{X}Y})^T(\Sigma_{\boldsymbol{XX}})^{-1}(\boldsymbol{X} - \mu_{\boldsymbol{X}})$$

**Bivariate case:** $M = 1, X(1) = X$

- $\Sigma_{XX} = \sigma_X^2$,

- $\Sigma_{XY} = \rho\sigma_X\sigma_Y$, where $\rho \equiv \dfrac{\Sigma_{XY}}{\sigma_X\sigma_Y}$ is the correlation coefficient of $Y$ and $X$.

In this case,

$$\begin{aligned} \widehat{Y} = \mathbf{E}[Y|X] &= \mu_Y + \frac{\rho\sigma_Y}{\sigma_X}(X - \mu_X) \\ &= \underbrace{\left(\mu_Y - \frac{\rho\sigma_Y}{\sigma_X}\mu_X\right)}_{h_0} + \underbrace{\left(\frac{\rho\sigma_Y}{\sigma_X}\right)}_{h_1} X \end{aligned}$$

We can observe that $\widehat{Y}$ is linear, i.e. is the LMMSEE $\hat{Y} = \widehat{Y}$ in the bivariate case. This is also true in the general multivariate Gaussian case. In fact,

$\hat{Y} = \widehat{Y}$ if, and only if, $[Y, X(1), \ldots, X(M)]^T$ is a Gaussian random vector.

## 2.3. Time-discrete Wiener filters

- ***Problem:***

  Estimation of a WSS random sequence $Y(n)$ based on the observation of another sequence $X(n)$. Without loss of generality we assume that $\mathbf{E}[Y(n)] = \mathbf{E}[X(n)] = 0$.

  The goodness of the estimator $\hat{Y}(n)$ is described by the MSE

$$\mathbf{E}[(\hat{Y}(n) - Y(n))^2].$$

  We distinguish between two cases:

  - **Prediction:**

    $\hat{Y}(n)$ depends on one or several past observations of $X(n)$ only, i.e.

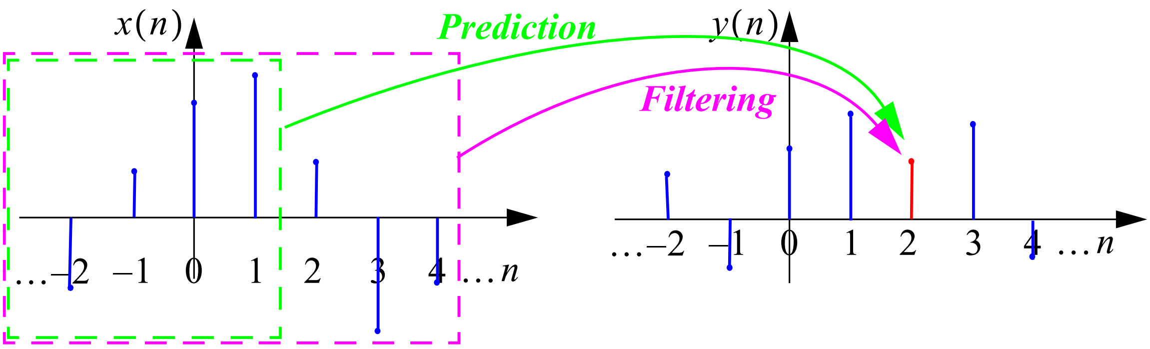
$$\hat{Y}(n) = \hat{Y}(X(n_1), X(n_2), \ldots) \text{ with } n_1, n_2, \ldots < n$$

  - **Filtering:**

    $\hat{Y}(n)$ depends on the present observation and/or one or many future observation(s) of $X(n)$, i.e.

$$\hat{Y}(n) = \hat{Y}(X(n_1), X(n_2), \ldots) \text{ where at least one } n_i \geq n$$

    If all $n_i \leq n$, the filter is **causal** otherwise it is **noncausal**.




**Typical application:** WSS signal embedded in additive white noise

$$X(n) = Y(n) + W(n) .$$

where,

- $W(n)$ is a white noise sequence,
- $Y(n)$ is a WSS process
- $Y(n)$ and $W(n)$ are uncorrelated.

However, the theoretical treatment is more general as shown below.

### 2.3.1. Noncausal Wiener filters

- ***Linear Minimum Mean Squared Error Filter***
  We seek a linear filter

$$\hat{Y}(n) = \sum_{m=-\infty}^{\infty} h(m)X(n-m) = h(n)*X(n)$$

which minimizes the MSE $\mathbf{E}[(\hat{Y}(n) - Y(n))^2]$.
Such a filter exists. It is called a **Wiener filter** in honour of his inventor.

- ***Orthogonality principle (time domain):***
  The coefficients of a Wiener filter satisfy the conditions:

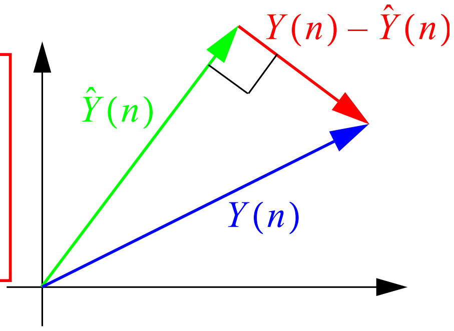
$$\mathbf{E}[(Y(n) - \hat{Y}(n))X(n-k)] =$$

$$= \mathbf{E}\left[\left(Y(n) - \sum_{m=-\infty}^{\infty} h(m)X(n-m)\right)X(n-k)\right] = 0, \quad k = \ldots, -1, 0, 1, \ldots$$

It follows from these identities (see also (2.3)) that

$$\mathbf{E}[(Y(n) - \hat{Y}(n))\hat{Y}(n)] = 0$$

$$\mathbf{E}[(Y(n) - \hat{Y}(n))^2] = \mathbf{E}[Y(n)^2] - \mathbf{E}[\hat{Y}(n)^2] = \mathbf{E}[(Y(n) - \hat{Y}(n))Y(n)]$$




With the definitions

$$R_{XX}(k) \equiv \mathbf{E}[X(n)X(n+k)], \; R_{XY}(k) \equiv \mathbf{E}[X(n)Y(n+k)]$$

we can recast the orthogonality conditions as follows:

$$R_{XY}(k) = \sum_{m=-\infty}^{\infty} h(m)R_{XX}(k-m) \qquad k = \ldots, -1, 0, 1, \ldots$$

$$R_{XY}(k) = h(k)*R_{XX}(k) \qquad \text{Wiener-Hopf equation}$$

- ***Orthogonality principle (frequency domain):***

$$S_{XY}(f) = H(f)S_{XX}(f)$$

where

$$S_{XY}(f) \equiv F\{R_{XY}(k)\}$$

- ***Transfer function of the Wiener filter:***

$$H(f) = \frac{S_{XY}(f)}{S_{XX}(f)}$$

- ***MSE of the Wiener filter (time-domain formulation):***

$$\mathbf{E}[(Y(n) - \hat{Y}(n))^2] = \sigma_Y^2 - \sum_{m=-\infty}^{\infty} h(m)R_{XY}(m)$$

***Proof:***

$$\mathbf{E}[(Y(n) - \hat{Y}(n))^2] = \mathbf{E}[Y(n)^2] - \mathbf{E}[\hat{Y}(n)Y(n)]$$

□

- ***MSE of the Wiener filter (frequency-domain formulation):***
We can rewrite the above identity as:

$$\mathbf{E}[(Y(n) - \hat{Y}(n))^2] = R_{YY}(0) - \sum_{m=-\infty}^{\infty} h(m)R_{YX}(-m)$$

$\mathbf{E}[(Y(n) - \hat{Y}(n))^2]$ is the value $p(0)$ of the sequence

$$p(k) = R_{YY}(k) - \sum_{m=-\infty}^{\infty} h(m) R_{YX}(k-m)$$

$$= R_{YY}(k) - h(k) * R_{YX}(k)$$

$\circ$
$\bullet$

$$P(f) = S_{YY}(f) - H(f) S_{YX}(f) = S_{YY}(f) - \frac{|S_{XY}(f)|^2}{S_{XX}(f)}$$

Hence,

$$\mathbf{E}[(Y(n) - \hat{Y}(n))^2] = p(0) = \int_{-1/2}^{1/2} P(f) df$$

$$\boxed{\mathbf{E}[(Y(n) - \hat{Y}(n))^2] = \int_{-1/2}^{1/2} \left[ S_{YY}(f) - \frac{|S_{XY}(f)|^2}{S_{XX}(f)} \right] df}$$

### 2.3.2. Causal Wiener filters

**A. $X(n)$ is a white noise.**

We first assume that $X(n)$ is a white noise with unit variance, i.e.
$\mathbf{E}[X(n)X(n+k)] = \delta(k)$.

- ***Derivation of the causal Wiener filter from the noncausal Wiener filter:***
  Let us consider the noncausal Wiener filter

$$\hat{Y}(n) = \sum_{m=-\infty}^{\infty} h(m) X(n-m)$$

whose transfer function is given by

$$H(f) = \frac{S_{XY}(f)}{S_{XX}(f)} = S_{XY}(f).$$

Then, the causal Wiener filter $\hat{Y}_c(n)$ results by cancelling the noncausal part of the non-causal Wiener filter:

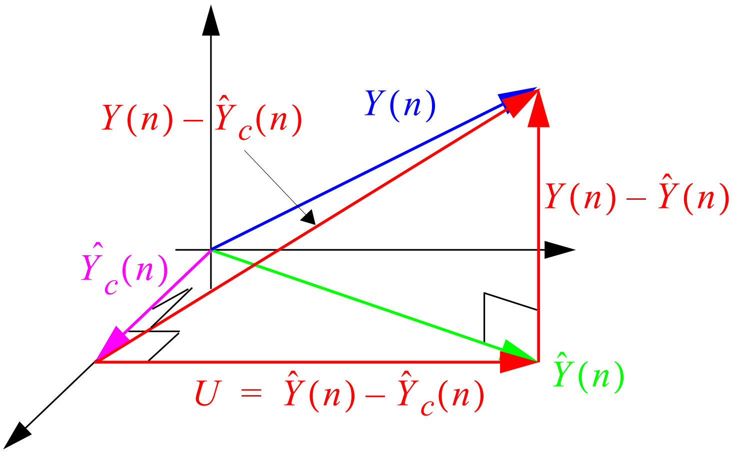
$$\boxed{\hat{Y}_c(n) = \sum_{m=0}^{\infty} h(m) X(n-m)}$$

***Sketch of the proof:***

$\hat{Y}(n)$ can be written as

$$\hat{Y}(n) = \underbrace{\sum_{m=-\infty}^{-1} h(m)X(n-m)}_{\equiv U} + \underbrace{\sum_{m=0}^{\infty} h(m)X(n-m)}_{V \equiv \hat{Y}_c(n)}$$

Because $X(n)$ is a white noise, the causal part $V = \hat{Y}_c(n)$ and the noncausal part $U = \hat{Y}(n) - \hat{Y}_c(n)$ of $\hat{Y}(n)$ are orthogonal. It follows from this property



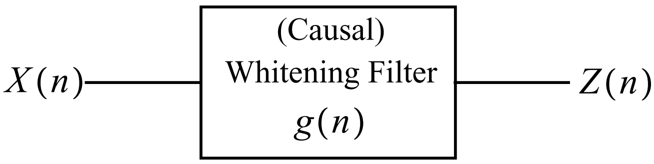

that $\hat{Y}_c(n)$ and $Y(n)$ are orthogonal, i.e. that $\hat{Y}_c(n)$ minimizes the MSE within the class of linear causal estimators.

□

**B.** $X(n)$ **is an arbitrary WSS process whose spectrum satisfies the Paley-Wiener condition.**

Usually, the above truncation procedure to obtain $\hat{Y}_c(n)$ does not apply because $U$ and $V$ are correlated and therefore not orthogonal in the general case.

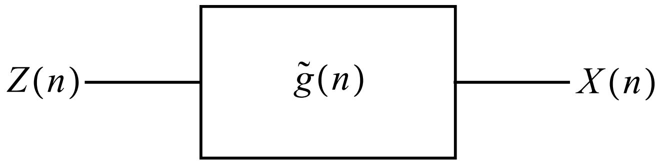
- ***Causal whitening filter:***
  However, we can show (see the Spectral Decomposition Theorem below) that provided $S_{XX}(f)$ satisfies the Paley-Wiener condition (see below) then $X(n)$ can be **converted into an equivalent white noise sequence** $Z(n)$ **with unit variance** by filtering it with an appropriate causal filter $g(n)$,




$$\mathbf{E}[Z(n)Z(n+k)] = \delta(k)$$

This operation is called **whitening** and the filter $g(n)$ is called a **whitening filter**.

equivalent $\equiv$ there exists another causal filter $\tilde{g}(n)$ so that $X(n) = \tilde{g}(n) * Z(n)$:




Notice that if

$$G(f) \equiv F\{g(n)\}$$

$$\tilde{G}(f) \equiv F\{\tilde{g}(n)\}$$

then

$$|G(f)|^2 = S_{XX}(f)^{-1}$$

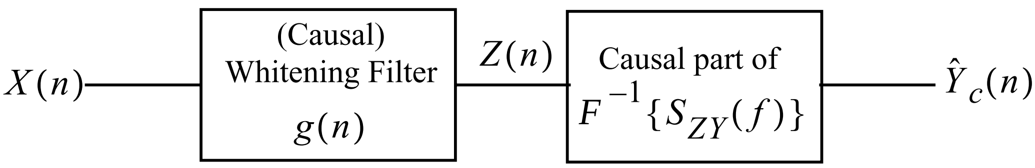
$$|\tilde{G}(f)|^2 = S_{XX}(f) \qquad (2.6)$$

We shall see that a whitening filter exists such that

$$\tilde{G}(f) = G(f)^{-1}$$

- ***Causal Wiener filter***

  Making use of the result in Part A, the block diagram of the causal Wiener filter is



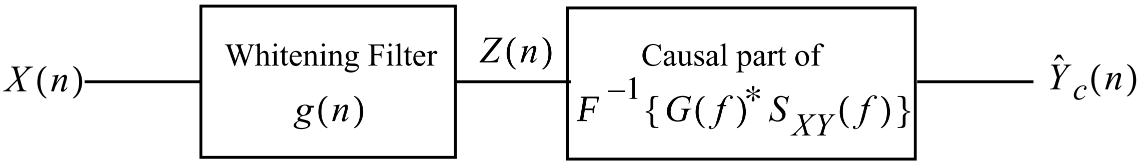

$S_{ZY}(f)$ is obtained from $S_{XY}(f)$ according to

$$\boxed{S_{ZY}(f) = G(f)^* S_{XY}(f)}$$

***Proof:***

$\square$

Hence, the block diagram of the causal Wiener filter is:




- ***Spectral Decomposition Theorem:***

  Let $S_{XX}(f)$ satisfies the so-called **Paley-Wiener condition**:

$$\int_{-1/2}^{1/2} \log(S_{XX}(f))df > -\infty$$

Then $S_{XX}(f)$ can be written as

$$S_{XX}(f) = G(f)^+ G(f)^-$$

with $G(f)^+$ and $G(f)^-$ satisfying

$$\left|G(f)^+\right|^2 = \left|G(f)^-\right|^2 = S_{XX}(f).$$

Moreover, the sequences

$$g(n)^+ \equiv F^{-1}\{G(f)^+\}$$
$$g(n)^- \equiv F^{-1}\{G(f)^-\}$$
$$g^{-1}(n)^+ \equiv F^{-1}\{1/G(f)^+\}$$
$$g^{-1}(n)^- \equiv F^{-1}\{1/G(f)^-\}$$

satisfy

$$g(n)^+ = g^{-1}(n)^+ = 0 \qquad n < 0 \qquad \text{Causal sequences}$$
$$g(n)^- = g^{-1}(n)^- = 0 \qquad n > 0 \qquad \text{Anticausal sequences}$$

- ***Whitening filter (cont'd):***
  The sought whitening filter used to obtained $Z(n)$ is

$$g(n) = g^{-1}(n)^{+}$$

and

$$\tilde{g}(n) = g(n)^{+}.$$

It can be easily verified that both sequences satisfy the identities in (2.6).

### 2.3.3. Finite Wiener filters

- ***Finite linear filter:***

$$\hat{Y}(n) = \sum_{m=-M_1}^{M_2} h(m) X(n-m)$$

- ***Wiener-Hopf equation:***
  By applying the orthogonality principle we obtain the Wiener-Hopf system of equations:

$$\Sigma_{\boldsymbol{XY}} = \Sigma_{\boldsymbol{XX}} \boldsymbol{h}$$

where

$$\boldsymbol{h} \equiv [h(-M_1), \ldots, h(M_2)]^T$$

and

$$\Sigma_{\boldsymbol{XY}} \equiv [R_{XY}(-M_1), \ldots, R_{XY}(M_2)]^T$$

$$\Sigma_{\boldsymbol{XX}} \equiv$$

$$\equiv \begin{bmatrix} R_{XX}(0) & R_{XX}(1) & R_{XX}(2) & \ldots & R_{XX}(M_1+M_2) \\ R_{XX}(1) & R_{XX}(0) & R_{XX}(1) & \ldots & R_{XX}(M_1+M_2-1) \\ R_{XX}(2) & R_{XX}(1) & R_{XX}(0) & \ldots & R_{XX}(M_1+M_2-2) \\ \ldots & \ldots & \ldots & \ldots & \ldots \\ R_{XX}(M_1+M_2) & R_{XX}(M_1+M_2-1) & R_{XX}(M_1+M_2-2) & \ldots & R_{XX}(0) \end{bmatrix}$$

Coefficient vector of the finite Wiener filter:

$$\boldsymbol{h} = (\Sigma_{\boldsymbol{XX}})^{-1} \Sigma_{\boldsymbol{XY}}$$

provided $\Sigma_{\boldsymbol{XX}}$ is invertible.